BRYN MAWR COLLEGE MONOGRAPHS

REPRINT SERIES, Vol. 4., ~~No. 4~~

CONTRIBUTIONS FROM THE MATHEMATICAL AND PHYSICAL DEPARTMENTS

BRYN MAWR, PENNA., U. S. A.
April 1904

Table of Contents.

On the Imprimitive Substitution Groups of Degree Fifteen and the Primitive Substitution Groups of Degree Eighteen.

By Emilie Norton Martin.

The following work is, with some slight modifications, the same as that of which an abstract was presented at the summer meeting of the American Mathematical Society in 1899. With regard to the imprimitive groups of degree fifteen, which form the subject matter of the first part of this paper, it should be stated that I have added two new groups to the list as originally presented, namely, the groups with five systems of imprimitivity simply isomorphic to the alternating and symmetric groups of degree 5, and that Dr. Kuhn reported at the February meeting of the Society, 1900, that he had carried the investigation further, adding 28 to the 70 groups that I succeeded in finding.

In the second part of this paper the determination of the primitive groups of degree 18 depends to a great extent upon the lists of transitive groups of lower degrees already determined. Any new discovery of groups of degree less than 18 would necessitate an examination of such groups to determine whether they can be combined with others in such a way as to generate a primitive group of degree 18. This list, therefore, cannot claim to be absolutely complete, since omissions are always possible.

Imprimitive Substitution Groups of Degree Fifteen.

Every imprimitive group contains a self-conjugate intransitive subgroup consisting of all the operations that interchange the elements of the systems of imprimitivity among themselves without interchanging the systems. Therefore, the problem of the determination of all imprimitive groups of degree 15 falls into two parts: 1st, the determination of all intransitive groups of degree 15

capable of becoming the self-conjugate subgroups of such imprimitive groups; 2d, the determination of substitutions that will interchange the systems of imprimitivity and at the same time fulfill other conditions depending upon the particular group under discussion. The intransitive self-conjugate subgroup is called for shortness the *head*, the remaining substitutions of the imprimitive group are designated as the *tail*, a terminology that has been adopted by Dr. G. A. Miller in his papers on imprimitive groups.

The elements of an imprimitive group of degree 15 may fall into three systems of five elements each, or into five systems of three each. For the first of these cases, certain theorems given by Dr. G. A. Miller (Quar. Jour. Math., vol. XXVIII, 1896) are useful. With a slight modification in notation in order to adapt them to the notation of this paper, they are as follows, where G^1 represents a group in the elements with index 1, while G^2 and G^3 represent precisely the same group in the elements with indices 2 and 3.

THEOREM I.—*All the substitutions that can be used to construct tails are*

$$(a_1^1a_2^1 \dots a_n^1) \text{ all } (a_1^2a_2^2 \dots a_n^2) \text{ all } (a_1^3a_2^3 \dots a_n^3) \text{ all}$$
$$\{(a_1^1a_1^2a_1^3 . a_2^1a_2^2a_2^3 . \dots . a_n^1a_n^2a_n^3),\ (a_1^1a_1^2 . a_2^1a_2^2 . \dots . a_n^1a_n^2)\}$$
$$- (a_1^1a_2^1 \dots a_n^1) \text{ all } (a_1^2a_2^2 \dots a_n^2) \text{ all } (a_1^3a_2^3 \dots a_n^3) \text{ all.}$$

THEOREM II.—*If* $G^1 = (a_1^1a_2^1 \dots a_n^1)$ all, *there are three imprimitive groups with the common head* $(G^1G^2G^3)$ pos, *and two with the common head* G^1 pos G^2 pos G^3 pos $+$ G^1 neg G^2 neg G^3 neg.

THEOREM III.—*If* $G^1 = (a_1^1a_2^1 \dots a_n^1)$ pos, *there are three imprimitive groups with the common head* $G^1G^2G^3$, *and three with the common head* $(G^1G^2G^3)_{1,1,1}$.

THEOREM IV.—*If the head is* $G^1G^2G^3$, *there is only one group which corresponds to* (abc) cyc.

The possible heads for these groups are got either by the direct multiplication of transitive groups of degree 5 in the three systems of elements, or by the establishment of isomorphic relations between such groups.

The transitive groups of degree 5 are five in number, and fall naturally into two categories, the first containing the symmetric group and its self-conjugate subgroup, the alternating group, the second containing the metacyclic group, together with its two self-conjugate transitive subgroups. These five groups are represented respectively by

$$(a_1a_2a_3a_4a_5) \text{ all},\ (a_1a_2a_3a_4a_5) \text{ pos},\ (a_1a_2a_3a_4a_5)_{20},\ (a_1a_2a_3a_4a_5)_{10},\ (a_1a_2a_3a_4a_5)_5 .$$

From the first two groups come the following heads:

I. $(a_1^1a_2^1a_3^1a_4^1a_5^1)$ all $(a_1^2a_2^2a_3^2a_4^2a_5^2)$ all $(a_1^3a_2^3a_3^3a_4^3a_5^3)$ all $= H_{1728000}$.

II. $\{(a_1^1a_2^1a_3^1a_4^1a_5^1)$ all $(a_1^2a_2^2a_3^2a_4^2a_5^2)$ all $(a_1^3a_2^3a_3^3a_4^3a_5^3)$ all$\}$ pos $= H_{864000}$.

III. $(a_1^1a_2^1a_3^1a_4^1a_5^1)$ pos $(a_1^2a_2^2a_3^2a_4^2a_5^2)$ pos $(a_1^3a_2^3a_3^3a_4^3a_5^3)$ pos
$+ (a_1^1a_2^1a_3^1a_4^1a_5^1)$ neg $(a_1^2a_2^2a_3^2a_4^2a_5^2)$ neg $(a_1^3a_2^3a_3^3a_4^3a_5^3)$ neg $= H_{432000}$.

IV. $(a_1^1a_2^1a_3^1a_4^1a_5^1)$ pos $(a_1^2a_2^2a_3^2a_4^2a_5^2)$ pos $(a_1^3a_2^3a_3^3a_4^3a_5^3)$ pos $= H_{216000}$.

V. $(a_1^1a_2^1a_3^1a_4^1a_5^1 \, . \, a_1^2a_2^2a_3^2a_4^2a_5^2 \, . \, a_1^3a_2^3a_3^3a_4^3a_5^3)$ all $= H_{120}$.

VI. $(a_1^1a_2^1a_3^1a_4^1a_5^1 \, . \, a_1^2a_2^2a_3^2a_4^2a_5^2 \, . \, a_1^3a_2^3a_3^3a_4^3a_5^3)$ pos $= H_{60}$.

From the three remaining groups come the heads :

VII. $(a_1^1a_2^1a_3^1a_4^1a_5^1)_{20}\,(a_1^2a_2^2a_3^2a_4^2a_5^2)_{20}\,(a_1^3a_2^3a_3^3a_4^3a_5^3)_{20} = H_{8000}$.

VIII. $\{(a_1^1a_2^1a_3^1a_4^1a_5^1)_{20}\,(a_1^2a_2^2a_3^2a_4^2a_5^2)_{20}\,(a_1^3a_2^3a_3^3a_4^3a_5^3)_{20}\}$ pos $= H_{4000}$.

IX. $\{(a_1^1a_2^1a_3^1a_4^1a_5^1)_{20}\,(a_1^2a_2^2a_3^2a_4^2a_5^2)_{20}\,(a_1^3a_2^3a_3^3a_4^3a_5^3)_{20}\}_{10,\,10,\,10} = H_{2000}$.

X. $(a_1^1a_2^1a_3^1a_4^1a_5^1)_{10}\,(a_1^2a_2^2a_3^2a_4^2a_5^2)_{10}\,(a_1^3a_2^3a_3^3a_4^3a_5^3)_{10} = H_{1000}$.

XI. $\{(a_1^1a_2^1a_3^1a_4^1a_5^1)_{20}\,(a_1^2a_2^2a_3^2a_4^2a_5^2)_{20}\,(a_1^3a_2^3a_3^3a_4^3a_5^3)_{20}\}_{5,\,5,\,5} = H_{500}$.

XII. $\{(a_1^1a_2^1a_3^1a_4^1a_5^1)_{10}\,(a_1^2a_2^2a_3^2a_4^2a_5^2)_{10}\,(a_1^3a_2^3a_3^3a_4^3a_5^3)_{10}\}_{5,\,5,\,5} = H_{250}$.

XIII. $(a_1^1a_2^1a_3^1a_4^1a_5^1)$ cyc $(a_1^2a_2^2a_3^2a_4^2a_5^2)$ cyc $(a_1^3a_2^3a_3^3a_4^3a_5^3)$ cyc $= H_{125}$.

XIV. $(a_1^1a_2^1a_3^1a_4^1a_5^1 \, . \, a_1^2a_2^2a_3^2a_4^2a_5^2 \, . \, a_1^3a_2^3a_3^3a_4^3a_5^3)_{20} = H_{20}$.

XV. $(a_1^1a_2^1a_3^1a_4^1a_5^1 \, . \, a_1^2a_2^2a_3^2a_4^2a_5^2 \, . \, a_1^3a_2^3a_3^3a_4^3a_5^3)_{10} = H_{10}$.

XVI. $(a_1^1a_2^1a_3^1a_4^1a_5^1 \, . \, a_1^2a_2^2a_3^2a_4^2a_5^2 \, . \, a_1^3a_2^3a_3^3a_4^3a_5^3)$ cyc. $= H_5$.

The groups corresponding to these heads may be isomorphic either to $(a^1a^2a^3)$ cyc or to $(a^1a^2a^3)$ all. To generate a group isomorphic to $(a^1a^2a^3)$ cyc a substitution with the following properties must be added to the head: it must have its cube in the head, it must interchange all three systems, and it must transform the head into itself. Calling the group so found G, the groups isomorphic to $(a^1a^2a^3)$ all may be found by combining with G any substitution that has its square in the head, that interchanges two of the systems leaving the third unaffected, and that transforms H into itself, and G into itself.

As all the heads given above are symmetric in the three sets of elements, each head furnishes two groups by means of the symmetrically formed substitutions

$$s = a_1^1a_1^2a_1^3 \, . \, a_2^1a_2^2a_2^3 \, . \, a_3^1a_3^2a_3^3 \, . \, a_4^1a_4^2a_4^3 \, . \, a_5^1a_5^2a_5^3 \, , \quad t = a_1^1a_1^2 \, . \, a_2^1a_2^2 \, . \, a_3^1a_3^2 \, . \, a_4^1a_4^2 \, . \, a_5^1a_5^2 \, .$$

The letters s and t are used throughout this section of the paper to denote these particular substitutions, other substitutions fulfilling the same conditions being denoted by the same letters with suffixes.

According to Theorem I, any s_a or t_a must be the product of some substitution, σ_a, of the most general head, $H_{1728900}$, by s or t. Therefore σ_a must be a substitution of a subgroup of $H_{1728000}$ that contains the special H under consideration as a self-conjugate subgroup.

We may now proceed to the determination of the groups to be derived from the various heads taken in order.

I. $H_{1728000}$ gives us, according to Theorem I, only the two groups,

$$\{H_{1728000},\ s,\}\ \text{of order}\ 5184000_1,$$
$$\text{and}\ \{H_{1728000},\ s,\ t\}\ \text{of order}\ 10368000.$$

II. H_{864000} gives us, in accordance with Theorem II, three distinct groups. Of these, two are the groups,

$$\{H_{864000},\ s\}\ \text{of order}\ 2592000_1,$$
$$\{H_{864000},\ s,\ t\}\ \text{of order}\ 5184000_2,$$

A σ that transforms the head into itself without belonging in the head is $\sigma = a_1^1 a_2^1$. This cannot be combined with s, as $(\sigma s)^3$ is an odd substitution; it may, however, be combined with t. The remaining group is therefore

$$\{H_{864000},\ s,\ a_1^1 a_2^1 . t\}\ \text{of order}\ 5184000_3.$$

Of these two groups of order 5184000, the first contains both odd and even substitutions, the second only even.

III. H_{432000} gives, by Theorem II, the two groups

$$\{H_{432000},\ s\}\ \text{of order}\ 1296000_1,$$
$$\{H_{432000},\ s,\ t\}\ \text{of order}\ 2592000_2.$$

IV. H_{216000} gives us, by Theorem III, three distinct groups. $\sigma = a_1^1 a_2^1$ transforms the head into itself, but when combined with s it gives an odd substitution whose cube cannot be found in the head. The substitution σt furnishes us however, with a new t_a. The three groups are, therefore,

$$\{H_{216000},\ s\}\ \text{of order}\ 648000,$$
$$\{H_{216000},\ s,\ t\}\ \text{of order}\ 1296000_2,$$
$$\{H_{216000},\ s,\ a_1^1 a_2^1 . t\}\ \text{of order}\ 1296000_3.$$

The two groups of order 1296000 are distinct, since the one contains both odd and even substitutions, the other only even.

V. H_{120} is not contained self-conjugately in any larger subgroup of $H_{1728000}$, therefore only the two following groups can be formed from it:

$$\{H_{120}, s\} \text{ of order } 360_1,$$
$$\{H_{120}, s, t\} \text{ of order } 720.$$

VI. H_{60} gives, in accordance with Theorem III, three groups:

$$\{H_{60}, s\} \text{ of order } 180,$$
$$\{H_{60}, s, t\} \text{ of order } 360_2,$$
$$\{H_{60}, s, a_1^1a_2^1 . a_1^2a_2^2 . a_1^3a_2^3 . t\} \text{ of order } 360_3.$$

The last of these groups consists entirely of even substitutions.

The remaining heads are all composed of substitutions of the type

$$v_{a^1}^{\alpha} u_{a^1}^{\alpha'} v_{a^2}^{\beta} u_{a^2}^{\beta'} v_{a^3}^{\gamma} u_{a^3}^{\gamma'}, \tag{1}$$

where $v_{a^1} = a_2^1a_3^1a_5^1a_4^1$, $u_{a^1} = a_1^1a_2^1a_3^1a_4^1a_5^1$, while v_{a^2}, u_{a^2}, v_{a^3}, u_{a^3} denote the same substitutions written in elements with the indices 2 and 3 respectively. The substitutions v_{a^1}, u_{a^1} generate the metacyclic group in the five elements with index 1, these substitutions being subject to the conditions

$$v_{a^1}^4 = 1, \quad u_{a^1}^5 = 1, \quad u_{a^1}^x v_{a^1}^y = v_{a^1}^y u_{a^1}^{2^y x}.$$

The most general s_a is given by

$$s_a = v_{a^1}^{a} u_{a^1}^{a_1} v_{a^2}^{b} u_{a^2}^{b_1} v_{a^3}^{c} u_{a^3}^{c_1} s. \tag{2}$$

From this we find

$$s_a^3 = v_{a^1}^{a+b+c} u_{a^1}^{\lambda} v_{a^2}^{a+b+c} u_{a^2}^{\mu} v_{a^3}^{a+b+c} u_{a^3}^{\nu}, \tag{3}$$

where

$$\left.\begin{aligned} \lambda &= 2^c(2^b a_1 + b_1) + c_1, \\ \mu &= 2^a(2^c b_1 + c_1) + a_1, \\ \nu &= 2^b(2^a c_1 + a_1) + b_1. \end{aligned}\right\} \tag{4}$$

Transformation of the general substitution (1) by s_a gives us

$$s_a^{-1} v_{a^1}^{\alpha} u_{a^1}^{\alpha'} v_{a^2}^{\beta} u_{a^2}^{\beta'} v_{a^3}^{\gamma} u_{a^3}^{\gamma'} s_a = v_{a^1}^{\gamma} u_{a^1}^{\nu} v_{a^2}^{\alpha} u_{a^2}^{\lambda} v_{a^3}^{\beta} u_{a^3}^{\mu}, \tag{5}$$

where

$$\left.\begin{aligned} \lambda &= a_1 + 2^a\alpha' - 2^\alpha a_1, \\ \mu &= b_1 + 2^b\beta' - 2^\beta b_1, \\ \nu &= c_1 + 2^c\gamma' - 2^\gamma c_1, \end{aligned}\right\} \tag{6}$$

The general substitution of the group $G = \{H, s_a\}$ is

$$T = s_a^r v_{a^1}^{\alpha} u_{a^1}^{\alpha'} v_{a^2}^{\beta} u_{a^2}^{\beta'} v_{a^3}^{\gamma} u_{a^3}^{\gamma'}. \tag{7}$$

The most general t_β is given by

$$t_\beta = v_{a^1}^{a_2} u_{a^1}^{a_3} v_{a^2}^{b_2} u_{a^2}^{b_3} v_{a^3}^{c_2} u_{a^3}^{c_3} t. \tag{8}$$

Upon squaring this substitution, we get

$$t_\beta^2 = v_{a^1}^{a_2 + b_2} u_{a^1}^{\lambda} v_{a^2}^{a_2 + b_2} u_{a^2}^{\mu} v_{a^3}^{2c_2} u_{a^3}^{\nu}, \tag{9}$$

where

$$\left.\begin{aligned} \lambda &= 2^{b_2} a_3 + b_3, \\ \mu &= 2^{a_2} b_3 + a_3, \\ \nu &= 2^{c_2} c_3 + c_3 . \end{aligned}\right\} \tag{10}$$

On transforming the general substitution T by the general t_β, we have, after a straight-forward calculation, the following expression for the case $x = 1$:

$$t_\beta^{-1} T_{x=1} t_\beta = s^2 v_{a^1}^{-a_2 + b_2 + \beta + a} u_{a^1}^{\lambda} v_{a^2}^{a_2 - c_2 + \alpha + c} u_{a^2}^{\mu} v_{a^3}^{-b_2 + c_2 + \gamma + b} u_{a_3}^{\nu}, \tag{11}$$

where

$$\left.\begin{aligned} \lambda &= -\, 2^{b_2 - a_2 + \beta + a} a_3 + 2^{\beta + b_2} a_1 + 2^{b_2}\beta' + b_3, \\ \mu &= -\, 2^{a_2 - c_2 + \alpha + c} c_3 + 2^{\alpha + a_2} c_1 + 2^{a_2}\alpha' + a_3, \\ \nu &= -\, 2^{c_2 - b_2 + \gamma + b} b_3 + 2^{\gamma + c_2} b_1 + 2^{c_2}\gamma' + c_3 . \end{aligned}\right\} \tag{12}$$

We may now return to the consideration of special groups.

VII. H_{8000} gives only the two groups formed with s and t, as any σ that might be used is already contained in this head. The groups are, therefore,

$$\{H_{8000},\, s\} \text{ of order } 24000_1,$$
$$\{H_{8000},\, s,\, t\} \text{ of order } 48000.$$

VIII. H_{4000} has the general substitution (1) subject to the condition $\alpha + \beta + \gamma \equiv 0 \pmod 2$. From (3), it is evident that s_a is subject to the condition $a + b + c \equiv 0 \pmod 2$. Therefore, s_a is already in the group generated by H_{4000} and by s, and there is only one group isomorphic to $(a^1 a^2 a^3)$ cyc. We find by (9) that every t_β has its square in the head, and by (11), that every t_β transforms the head into itself, therefore, we may take as a new t_β the simplest substitution for which $a_2 + b_2 + c_2 \equiv 1 \pmod 2$, viz.:

$$v_a t = a_1^1 a_1^2 \,.\, a_2^1 a_3^2 a_3^1 a_5^2 a_5^1 a_4^2 a_4^1 a_2^2 .$$

The three groups with this head are, therefore,

$$\{H_{4000},\, s\} \text{ of order } 12000_1,$$
$$\{H_{4000},\, s,\, t\} \text{ of order } 24000_2,$$
$$\{H_{4000},\, s,\, v_a t\} \text{ of order } 24000_3.$$

Of these groups the first and third consist of even substitutions, the second of even and odd.

IX. H_{2000} has the general substitution subject to the condition $\alpha \equiv \beta \equiv \gamma$ (mod 2). From (3) and (5), it is plain that every s_a can be used to generate a group of the kind required. The only possible form for the cofactor of s, if it is not to give the group generated by s and the head, is $v_{a^1}^a v_{a^2}^b v_{a^3}^c$, where a, b, c do not fulfill the condition $a \equiv b \equiv c$ (mod 2). The simplest form for such a cofactor, and a form to which all others reduce, is found by making two of the exponents vanish and the third become equal to 1, e. g., $s_1 = v_{a^1} s = a_1^1 a_1^2 a_1^3 \,.\, a_2^1 a_3^2 a_3^3 a_3^1 a_5^2 a_5^3 a_5^1 a_4^2 a_4^3 a_4^1 a_2^2 a_2^3$. Now, $s_1^4 = s_1 \,.\, v_{a^1} v_{a^2} v_{a^3}$ and $s_1^8 = s_1^2 \,.\, v_{a^1}^2 v_{a^2}^2 v_{a^3}^2$; we may, therefore, take s_1^4 as the s. in the place of s_1 and still have the same group. But $s_1^4 = (v_{a^2}^2 v_{a^3}^3)^{-1} s \, (v_{a^2}^2 v_{a^3}^3)_a$ therefore the group we have now found is merely the transformed of the group generated by s with respect to the substitution $v_{a^2}^2 v_{a^3}^3$. Consequently, there is but one group corresponding to the cyclic group of degree three.

If, in addition to the group given by t, we have a group given by t_β, then according to the relations derived from (11), $a_2 \equiv b_2 \equiv c_2$ (mod 2), i. e., the possible values of t_β are already present in the group generated with the help of t. The two imprimitive groups with this head are, therefore, the groups

$$\{H_{2000},\, s\} \text{ of order } 6000.$$
$$\{H_{2000},\, s,\, t\} \text{ of order } 12000_2.$$

In this, and all following work, the terms u in the cofactors of s and t are taken as unity, unless the contrary is expressly stated.

X. H_{1000} has its general substitution subject to the condition $\alpha \equiv \beta \equiv \gamma \equiv 0$ (mod 2). By Theorem IV, this head gives only one group isomorphic to (abc) cyc. If, in addition to the substitution t, there is a substitution t_β, the relations satisfied by the exponents of the v's in (11) reduces to $a_2 \equiv b_2 \equiv c_2$ (mod 2). We have, therefore, two distinct groups according as a_2 is even or odd. The three groups with this head are

$$\{H_{1000},\, s\} \text{ of order } 3000_1,$$
$$\{H_{1000},\, s,\, t\} \text{ of order } 6000_2,$$
$$\{H_{1000},\, s,\, v_{a^1} v_{a^2} v_{a^3} t\} \text{ of order } 6000_3.$$

XI. H_{500} subjects the general substitution to the conditions $\alpha = \beta = \gamma$, where $\alpha = 0, 1, 2, 3$. Since every substitution s_a satisfies the necessary conditions, the following independent types of s_a must be examined: $v_{a^1} s$, $v_{a^1}^2 s$, $v_{a^1}^3 s$, $v_{a^1} v_{a^2}^2 s$. The

fourth power of these substitutions is in every case the transformed of s with respect to some combination of the v's; therefore, they give nothing new. The possible forms for t_β are derived from the equation easily deducible from (11); $-a_2+b_2\equiv a_2-c_2\equiv -b_2+c_2$ (mod 4), which, taken in conjunction with the limited range of values of a_2, b_2, c_2, gives $a_2=b_2=c_2$. That is, every possible t_β is already included in the group generated by t. This head gives accordingly only the two groups,

$$\{H_{500},\, s\} \text{ of order } 1500_1,$$
$$\{H_{500},\, s,\, t\} \text{ of order } 3000_2.$$

XII. H_{250} subjects the general substitution (1) to the conditions $\alpha=\beta=\gamma\equiv 0$ (mod 2). To determine an s_a, we have from (3) the condition $a+b+c\equiv 0$ (mod 2). An examination of the four apparently distinct types of s_a, $v_1^2 s$, $v_1 v_2 s$, $v_1^3 v_2 s$, $v_1^3 v_2^3 s$, shows that just as in the last set of groups, these each give a group that can be derived from the group generated by s by means of an easy transformation.

The possible forms t_β must fulfill the conditions, deducible from (11), $-a_2+b_2\equiv -b_2+c_2\equiv -c_2+a_2\equiv 0$ (mod 2) and also $-a_2+b_2\equiv a_2-c_2$ (mod 4). These reduce to the simple condition $a_2=b_2=c_2$, which furnishes the substitution $t_\beta=v_{a^1}v_{a^2}v_{a^3}t$. This head gives therefore the three groups,

$$\{H_{250},\, s\} \text{ of order } 750_1,$$
$$\{H_{250},\, s,\, t\} \text{ of order } 1500_2,$$
$$\{H_{250},\, s,\, v_{a^1}v_{a^2}v_{a^3}\, t\} \text{ of order } 1500_3.$$

The second group alone contains odd substitutions.

XIII. H_{125} gives in accordance with Theorem IV only one group in which the systems are interchanged cyclically. The general substitution of this head is subject to the condition $\alpha=\beta=\gamma=0$. Applying this condition to (9) and (11) we find $a_2=b_2=c_2$, while a_2 lies under the further restriction of being even. Therefore we have in addition to t the substitution,

$$t_\beta=v_{a^1}^2 v_{a^2}^2 v_{a^3}^2\, t=a_1^1a_1^2.\, a_2^1a_5^2.\, a_3^1a_4^2.\, a_4^1a_3^2.\, a_5^1a_2^2.\, a_2^3a_5^3.\, a_3^3a_4^3.$$

The three groups given by this head are,

$$\{H_{125},\, s\} \text{ of order } 375,$$
$$\{H_{125},\, s,\, t\} \text{ of order } 750_2,$$
$$\{H_{125},\, s,\, v_{a^1}^2 v_{a^2}^2 v_{a^3}^2\, t\} \text{ of order } 750_3.$$

XIV. H_{20} imposes upon the exponents of the general term the conditions

$\alpha = \beta = \gamma$, $\alpha' = \beta' = \gamma'$. Making use of this in (5) and (6) we find $2^a\alpha' \equiv 2^b\alpha' \equiv 2^c\alpha'$ (mod 5), which gives at once $a = b = c$. Using this latter equality in the equations that are deduced from (3) and (4) we find $a_1 = b_1 = c_1$ with the single exception of the case $a = 0$, where the equations become indeterminate, being satisfied by every value of a_1, b_1, c_1. An examination of all of the apparently independent sets of value for a_1, b_1, c_1 shows that in every case the group is transformable into that generated by s alone. In order to determine all substitutions t_β we use the equation, derived from (11), $-a_2 + b_2 \equiv a_2 - c_2 \equiv c_2 - b_2$ (mod. 4), from which follows at once $a_2 = b_2 = c_2$. From (12), by making use of the special case $\alpha = \beta = \gamma = 0$, can be derived the relations $-a_3 + b_3 \equiv -c_3 + a_3 \equiv -b + c_3$ (mod. 5); i, e, $a_3 = b_3 = c_3$. The only groups with this head are therefore the two groups,

$\{H_{20}, s\}$ of order 60_1,
$\{H_{20}, s, t\}$ of order 120.

XV. H_{10} has the general term (1) subject to the conditions $\alpha = \beta = \gamma \equiv 0$ (mod 2), $\alpha' = \beta' = \gamma'$. By precisely the same line of argument as that laid down in the preceding case we arrive at the conclusion $a = b = c$, $a_1 = b_1 = c_1$, $a_2 = b_2 = c_2$, $a_3 = b_3 = c_3$. In this work, too, the indeterminate values of a_1, b_1, c_1 require a careful examination that leads to no new group. From this head come, therefore, the three groups,

$\{H_{10}, s\}$ of order 30_1,
$\{H_{10}, s, t\}$ of order 60_2,
$\{H_{10}, s, v_{a^1} v_{a^2} v_{a^3} t\}$ of order 60_3.

Of these three groups the second alone involves odd substitutions.

XVI. H_5 imposes upon the general term the conditions $\alpha = \beta = \gamma = 0$, $\alpha' = \beta' = \gamma'$, By arguments similar to those used in the last two cases, with the further addition of the condition imposed by (3), $a + b + c \equiv 0$ (mod 4), we find $a = b = c = 0$, $a_1 = b_1 = c_1$. In the determination of t_β we see at once from (9) that c_2 must be even, while from (11) we find $a_2 = b_2 = c_2$, and from (12) $a_3 = b_3 = c_3$.

The groups given by this head are as follows:

$\{H_5, s\}$ of order 15,
$\{H_5, s, t\}$ of order 30_2,
$\{H_5, s, v_{a^1}^2 v_{a^2}^2 v_{a^3}^2 t\}$ of order 30_3.

Passing now to the case of five systems of three elements each, there are seven heads considered in this paper, six involving all the systems symmetrically, the remaining head being unity.

I. $(a_1^1 a_2^1 a_3^1)$ all $(a_1^2 a_2^2 a_3^2)$ all $(a_1^3 a_2^3 a_3^3)$ all $(a_1^4 a_2^4 a_3^4)$ all $(a_1^5 a_2^5 a_3^5)$ all $= H_{7776}$,
II. $\{H_{7776}\}$ pos $= H_{3888}$,
III. $\{H_{7776}\}_{3,\,3,\,3,\,3,\,3} = H_{486}$,
IV. $(a_1^1 a_2^1 a_3^1)$ pos $(a_1^2 a_2^2 a_3^2)$ pos $(a_1^3 a_2^3 a_3^3)$ pos $(a_1^4 a_2^4 a_3^4)$ pos $(a_1^5 a_2^5 a_3^5)$ pos $= H_{243}$,
V. $(a_1^1 a_2^1 a_3^1 \,.\, a_1^2 a_2^2 a_3^2 \,.\, a_1^3 a_2^3 a_3^3 \,.\, a_1^4 a_2^4 a_3^4 \,.\, a_1^5 a_2^5 a_3^5)$ all $= H_6$,
VI. $(a_1^1 a_2^1 a_3^1 \,.\, a_1^2 a_2^2 a_3^2 \,.\, a_1^3 a_2^3 a_3^3 \,.\, a_1^4 a_2^4 a_3^4 \,.\, a_1^5 a_2^5 a_3^5)$ cyc $= H_3$,
VII. Unity.

Denoting the system with index r by A_r, it is evident that these systems must be interchanged according to the five groups $(A_1A_2A_3A_4A_5)$ cyc, $(A_1A_2A_3A_4A_5)_{10}$, $(A_1A_2A_3A_4A_5)_{20}$, $(A_1A_2A_3A_4A_5)$ pos, $(A_1A_2A_3A_4A_5)$ all.

The order of procedure in each case is as follows:

1. If the group is to correspond to $(A_1A_2A_3A_4A_5)$ cyc, a substitution s must be found that will interchange the systems cyclically, transform the head into itself, and have its fifth power in the head. The imprimitive group so generated may be called G.

2. If the group is to correspond to $(A_1A_2A_3A_4A_5)_{10}$, it must contain G_1 as a self-conjugate subgroup. In addition, therefore, to the s of case 1, a substitution t must be found that will interchange four of the systems in two pairs, as $A_2A_5 \,.\, A_3A_4$, while leaving the remaining system unaltered, and that will, at the same time, transform the head into itself and G into itself. This substitution t must also have its square in the head. This imprimitive group shall be called G_2.

3. If the group is to correspond to $(A_1A_2A_3A_4A_5)_{20}$, it must contain both G_1 and G_2 as self-conjugate subgroups. In addition, therefore, to the s of case 1, a substitution u must be found interchanging four of the systems cyclically, according to $A_2A_3A_5A_4$ for instance, transforming G_1 and G_2 into themselves and having its fourth power in the head.

4. If the group is to correspond to $(A_1A_2A_3A_4A_5)$ pos, two substitutions, v and v', must be found corresponding to $A_1A_2A_3$ and $A_1A_4A_5$. These sub-

stitutions must, therefore, each interchange three systems, leaving two unaltered, they must have their cubes in the head, and must transform the head into itself. This group may be called G'.

5. If the group is to correspond to $(A_1A_2A_3A_4A_5)$ all, two substitutions, w and w', must be found corresponding to $A_1A_2A_3A_4$ and A_1A_5. G' is to be contained in this new group as a self-conjugate subgroup, therefore w and w' must transform the head into itself and G' into itself. The fourth power of w and the square of w' must both be contained in the head.

I. H_{7776} is the largest possible intransitive group with the given systems of intransitivity, and, consequently, only one group with this head corresponds to each of the transitive groups of degree 5. For each of these groups a substitution or pair of substitutions can be found fulfilling all required conditions and involving the elements of the systems symmetrically. A second set could be found only by multiplying this first step by some substitution belonging to the largest group that contains the head self-conjugately without interchanging any of the systems. But this group is the head itself. The required groups are, therefore, the following:

$$\begin{aligned}
&\{H_{7776},\, s\} \text{ of order } 38880,\\
&\{H_{7776},\, s,\, t\} \text{ of order } 77760,\\
&\{H_{7776},\, s,\, u\} \text{ of order } 155520,\\
&\{H_{7776},\, v,\, v'\} \text{ of order } 466560_1,\\
&\{H_{7776},\, w,\, w'\} \text{ of order } 933120,
\end{aligned}$$

where

$$\begin{aligned}
s &= a_1^1a_1^2a_1^3a_1^4a_1^5 \,.\, a_2^1a_2^2a_2^3a_2^4a_2^5 \,.\, a_3^1a_3^2a_3^3a_3^4a_3^5,\\
t &= a_1^2a_1^5 \,.\, a_1^3a_1^4 \,.\, a_2^2a_2^5 \,.\, a_2^3a_2^4 \,.\, a_3^2a_3^5 \,.\, a_3^3a_3^4,\\
u &= a_1^2a_1^3a_1^5a_1^4 \,.\, a_2^2a_2^3a_2^5a_2^4 \,.\, a_3^2a_3^3a_3^5a_3^4,\\
v &= a_1^1a_1^2a_1^3 \,.\, a_2^1a_2^2a_2^3 \,.\, a_3^1a_3^2a_3^3,\\
v' &= a_1^1a_1^4a_1^5 \,.\, a_2^1a_2^4a_2^5 \,.\, a_3^1a_3^4a_3^5,\\
w &= a_1^1a_1^2a_1^3a_1^4 \,.\, a_2^1a_2^2a_2^3a_2^4 \,.\, a_3^1a_3^2a_3^3a_3^4,\\
w' &= a_1^1a_1^5 \,.\, a_2^1a_2^5 \,.\, a_3^1a_3^5.
\end{aligned}$$

These letters shall be kept throughout this section of the paper to denote these symmetrically formed substitutions, other substitutions with corresponding properties being denoted by the same letters with suffixes.

II. H_{3888} gives only one group isomorphic to $(A_1 A_2 A_3 A_4 A_5)$ cyc, viz., the group generated by s. Any new s_a must have as cofactor an odd substitution belonging to H_{7776}, but the fifth power of such a substitution is not contained in the head. There are, however, two groups isomorphic to $(A_1 A_2 A_3 A_4 A_5)_{10}$, since both t and $t_a = a_1^1 a_2^1 . t$ fulfill the necessary conditions. The former generates a group G_{38880_2} containing only even substitutions, the latter generates a group G_{38880_3} containing both odd and even substitutions. There are likewise two groups isomorphic to $(A_1 A_2 A_3 A_4 A_5)_{20}$, one generated by u, the other by $a_1^1 a_2^1 . u$. The first of these groups contains odd substitutions, the second only even. G_{38880_2} is contained self-conjugately in both.

Only one group G' can be found for this head, as no new v_a or v'_a fulfills the necessary conditions. Such a substitution would necessarily be of the form σv or $\sigma v'$, where σ would belong to the group H_{7776}. If σ were even, the group so generated would be a repetition of the group generated by v and v'. If σ were odd, the cubes of σv, $\sigma v'$ would not be contained in the head.

Two groups can be found isomorphic to $(A_1 A_2 A_3 A_4 A_5)$ all, the substitutions w and w' generating one group, the substitutions $a_1^1 a_2^1 . w$, $a_1^1 a_2^1 . w'$ generating the other. This latter group contains only even substitutions.

From this head we have, therefore, derived eight groups:

$\{H_{3888}, s\}$ of order 19440,
$\{H_{3888}, s . t\}$ of order 38880_2,
$\{H_{3888}, s, a_1^1 a_2^1 . t\}$ of order 38880_3,
$\{H_{3888}, s, u\}$ of order 76660_2,
$\{H_{3888}, s, a_1^1 a_2^1 . u\}$ of order 76660_3,
$\{H_{3888}, v, v'\}$ of order 233280,
$\{H_{3888}, w, w'\}$ of order 466560_2,
$\{H_{3888}, a_1^1 a_2^1 . w, a_1^1 a_2^1 . w'\}$ of order 466560_3.

III. H_{486} furnishes us with only one group isomorphic to $(A_1 A_2 A_3 A_4 A_5)$ cyc, for an examination of the groups given by all possible types of substitutions s_a shows that each of these groups is merely the group generated by the help of s and transformed with respect to some easily discovered substitution. Moreover, there is but one group isomorphic to $(A_1 A_2 A_3 A_4 A_5)_{10}$, viz., that generated with the help of t. Any cofactor of t must be of one of the types $a_1^1 a_2^1$, $a_1^2 a_2^2 . a_1^5 a_2^5$, $a_1^1 a_2^1 . a_1^2 a_2^2 . a_1^5 a_2^5$, $a_1^2 a_2^2 . a_1^3 a_2^3 . a_1^4 a_2^4 . a_1^5 a_2^5$, but any t_a got by means of these, transforms

s into s^4 (a substitution not in the head). Precisely the same reasoning shows that there is only the one group isomorphic to $(A_1 A_2 A_3 A_4 A_5)_{20}$.

In addition to the group isomorphic to $(A_1 A_2 A_3 A_4 A_5)$ pos generated by means of the substitutions v and v', we must examine groups generated with the help of v_a and v'_a, substitutions which contain as cofactors of v, v' respectively the products of transposition, one transposition from each system. A number of these may be rejected at once, but we are left with the possible forms:

$$v_1 = a_1^1 a_2^1 \,.\, a_1^2 a_2^2 \,.\, v = a_1^1 a_2^2 a_1^3 \,.\, a_2^1 a_1^2 a_2^3 \,.\, a_3^1 a_3^2 a_3^3 \,,$$
$$v_2 = a_1^1 a_2^1 \,.\, a_1^4 a_2^4 \,.\, a_1^5 a_2^5 \,.\, v = a_1^1 a_2^2 a_2^3 \,.\, a_2^1 a_1^2 a_1^3 \,.\, a_1^4 a_2^4 \,.\, a_1^5 a_2^5 \,,$$
$$v'_1 = a_1^1 a_2^1 \,.\, a_1^4 a_2^4 \,.\, v' = a_1^1 a_2^4 a_1^5 \,.\, a_2^1 a_1^4 a_2^5 \,.\, a_3^1 a_3^4 a_3^5 \,,$$
$$v'_2 = a_1^1 a_2^1 \,.\, a_1^2 a_2^2 \,.\, a_1^3 a_2^3 \,.\, v' = a_1^1 a_2^4 a_2^5 \,.\, a_2^1 a_1^4 a_1^5 \,.\, a_1^2 a_2^2 \,.\, a_1^3 a_2^3 \,.$$

But, since v_1, v_2^4 are transformable into v, and v'_1, $v_2'^4$ are transformable into v', it is impossible to generate any group by means of any combination of these four substitutions excepting a group that can be transformed into the one generated by means of v and v'.

A similar examination of all groups isomorphic to $(A_1 A_2 A_3 A_4 A_5)$ all, shows that, in addition to the group generated with the help of w and w', there is one other group generated by means of $w_a = a_1^5 a_2^5 \,.\, w$ and w'.

From this head are therefore formed the six following groups:

$$\{H_{486}\,,\ s\} \text{ of order } 2430_1\,,$$
$$\{H_{486}\,,\ s\,,\ t\} \text{ of order } 4860_1\,,$$
$$\{H_{486}\,,\ s\,,\ u\} \text{ of order } 9720\,,$$
$$\{H_{486}\,,\ v\,,\ v'\} \text{ of order } 29160_1\,,$$
$$\{H_{486}\,,\ w\,,\ w'\} \text{ of order } 58320_1\,,$$
$$\{H_{436}\,,\ a_1^5 a_2^5 \,.\, w\,,\ w'\} \text{ of order } 58320_2.$$

IV. H_{243} gives one group isomorphic to $(A_1 A_2 A_3 A_4 A_5)$ cyc, by means of s. The only other permissible forms of s_a are of the type

$$s_1 = a_1^1 a_2^1 \,.\, a_1^2 a_2^2 \,.\, s\,, \quad s_2 = a_1^1 a_2^1 \,.\, a_1^2 a_2^2 \,.\, a_1^3 a_2^3 \,.\, a_1^4 a_2^4 \,.\, s\,.$$

But s_1 and s_2 are each the transformed of s with respect to some substitution that transforms the head into itself; therefore, there is only the one group of this type. On the other hand, there are two groups isomorphic to $(A_1 A_2 A_3 A_4 A_5)_{10}$, since both t and $t_1 = a_1^1 a_2^1 \,.\, a_1^2 a_2^2 \,.\, a_1^3 a_2^3 \,.\, a_1^4 a_2^4 \,.\, a_1^5 a_2^5 \,.\, t$ fulfill all necessary conditions and generate, one a group of even substitutions, the other a group containing odd

substitutions. There are also two groups isomorphic to $(A_1 A_2 A_3 A_4 A_5)_{20}$, one containing both odd and even substitutions, the other only even. These are generated respectively by means of u and of $u_1 = a_1^1 a_2^1 . a_1^2 a_2^2 . a_1^3 a_2^3 . a_1^4 a_2^4 . a_1^5 a_2^5 . u$, and each contains as a self-conjugate subgroup the group isomorphic to $(A_1 A_2 A_3 A_4 A_5)_{10}$ that consists entirely of even substitutions.

Only one group can be found isomorphic to $(A_1 A_2 A_3 A_4 A_5)$ pos, and this is the one formed by the help of v and v'. An examination of the various substitutions v_a and v'_a corresponding to various types of cofactor of v and v' shows that all groups formed by means of these substitutions are transformable into the one group.

On the other hand, we have two distinct groups corresponding to $(A_1 A_2 A_3 A_4 A_5)$ all, the one consisting of both odd and even substitutions and generated by the aid of w and w', the other consisting entirely of even substitutions and generated by the aid of $a_1^5 a_2^5 . w$ and $a_1^2 a_2^2 . w'$.

From this head we have, therefore, the eight following groups:

$\{H_{243}, s\}$ of order 1215,
$\{H_{243}, s, t\}$ of order 2430_2,
$\{H_{243}, s, a_1^1 a_2^1 . a_1^2 a_2^2 . a_1^3 a_2^3 . a_1^4 a_2^4 . a_1^5 a_2^5 . t\}$ of order 2430_3,
$\{H_{243}, s, u\}$ of order 4860_2,
$\{H_{243}, s, a_1^1 a_2^1 . a_1^2 a_2^2 . a_1^3 a_2^3 . a_1^4 a_2^4 . a_1^5 a_2^5 . u\}$ of order 4860_3,
$\{H_{243}, v, v'\}$ of order 14580,
$\{H_{243}, w, w'\}$ of order 29160_2,
$\{H_{243}, a_1^5 a_2^5 . w, a_1^2 a_2^2 . w'\}$ of order 29160_3.

V. H_6 furnishes one group corresponding to each transitive group of degree 5. These groups are generated respectively by the substitutions s, t, u, v, v', w, w', and can readily be seen to be indentical with those of orders 30_2, 60_2, 120, 360_2, 720 included among the groups with three systems of imprimitivity. An interchange of suffixes and indices in the one set of groups gives the generating substitution of the other set of groups.

VI. H_3 furnishes groups corresponding to the transitive groups of degree 5 by means of the substitutions s, t, u, v, v', w, w'. As in the last case, however, these correspond to the groups of orders 15, 30_1, 60_1, 180, 360_1 included in the groups with three systems of imprimitivity. By the use of the cofactor $\sigma = a_1^1 a_2^1$.

$a_1^2a_2^2 \cdot a_1^3a_2^3 \cdot a_1^4a_2^4 \cdot a_1^5a_2^5$ three more groups can be found generated respectively by the help of $t_1 = \sigma t$, $u_1 = \sigma u$, $w_1 = \sigma w$, $w_1' = \sigma w'$. These groups, however, are seen to be identical with those of orders 30_3, 60_3, and 360_3 included in the groups with three systems of imprimitivity. This head gives, therefore, no group essentially new.

VII. In the discussion of the head unity a useful theorem is the following given by Frobenius (Crelle t. CI, p. 287):

The average number of elements in all the substitutions of a group is $n - \alpha$, n being the degree of the group, and α the number of its transitive constituents.

The only transitive groups of degree 5 containing 15 as a factor of the order are the symmetric and alternating groups. We have therefore to find an imprimitive group of degree 15 with 5 systems of intransitivity simply isomorphic to the alternating (symmetric) group in 5 letters.

In determining the imprimitive group corresponding to $(A_1A_2A_3A_4A_5)$ pos, we make use of the following facts: (1) the 15 conjugate substitutions corresponding to terms of the type $A_1A_2 \cdot A_3A_4$ must be of degrees 12 or 14; (2) the 20 conjugate substitutions corresponding to terms of the type $A_1\ A_2\ A_3$ must be of degrees 9, 12, or 15; (3) the 24 conjugate substitutions corresponding to terms of the type $A_1A_2A_3A_4A_5$ must be of degree 15. It must, therefore, be possible to solve the equation

$$15\,(12 + 2\alpha) + 20\,(9 + 3\beta) + 24.15 = 14.60$$

where $\alpha = 0, 1$; $\beta = 0, 1, 2$. The only solution is $\alpha = 0$, $\beta = 2$.

Therefore the imprimitive group we are seeking contains among its substitutions 15 of degree 12 and order 2, 20 of degree 15 and order 3, 24 of degree 15 and order 5. Making use of the relations among the generating substitutions of such a group of order 60 as given in Burnside, Theory of Groups, p. 107, we find that the two substitutions corresponding to $A_1A_2A_3A_4A_5$, $A_1A_2A_3A_4^2$, substitutions which will generate $(A_1A_2A_3A_4A_5)$ pos, are respectively,

$$s = a_1^1a_1^2a_1^3a_1^4a_1^5 \cdot a_2^1a_2^2a_2^3a_2^4a_2^5 \cdot a_3^1a_3^2a_3^3a_3^4a_3^5,$$
$$\rho = a_1^1a_3^2 \cdot a_1^3a_2^4 \cdot a_2^1a_1^2 \cdot a_2^3a_3^4 \cdot a_3^1a_2^2 \cdot a_3^3a_1^4;$$

s and ρ are therefore the generating substitutions of an imprimitive group simply isomorphic to the alternating group of degree 5.

In determining a group simply isomorphic to $(A_1A_2A_3A_4A_5)$ all, we argue as before in regard to the various sets of conjugate substitutions. The 15 substitu-

tions corresponding to terms of the type $A_1A_2 \,.\, A_3A_4$ are of degrees 12 or 14, the 20 corresponding to the type $A_1A_2A_3$ are of degrees 9, 12, or 15, the 24 corresponding to the type $A_1A_2A_3A_4A_5$ are of degree 15, the 10 corresponding to the type A_1A_2 are of degrees 6, 8, 10, or 12; the 30 corresponding to the type $A_1A_2 A_3A_4$ are of degrees 12 or 14; the 20 corresponding to the type $A_1A_2A_3A_4A_5$ are of degree 15. The equation to be satisfied is therefore

$$15\,(12 + 2\alpha) + 20\,(9 + 3\beta) + 24.15 + 10\,(6 + 2\gamma) + 30\,(12 + 2\delta) + 20.15$$
$$= 14.120 \text{ where } \alpha = 0,\ 1\,;\ \beta = 0,\ 1,\ 2\,;\ \gamma = 0,\ 1,\ 2,\ 3\,;\ \delta = 0,\ 1.$$

The only solution is $\alpha = 0$, $\beta = 2$, $\gamma = 3$, $\delta = 1$. The substitutions $A_1A_2A_3A_4A_5$, $A_2A_3A_4A_5$, $A_1A_2A_3A_4$ will generate the group $(A_1A_2A_3A_4A_5)$ all, and corresponding to these as generators of the imprimitive group we have the three substitutions,

$$s = a_1^1a_1^2a_1^3a_1^4a_1^5 \,.\, a_2^1a_2^2a_2^3a_2^4a_2^5 \,.\, a_3^1a_3^2a_3^3a_3^4a_3^5,$$
$$\sigma = a_2^1a_3^1 \,.\, a_1^2a_1^3a_1^5a_1^4 \,.\, a_2^2a_3^2a_2^5a_3^4 \,.\, a_3^2a_2^3a_3^5a_2^4,$$
$$\rho = a_1^1a_3^2 \,.\, a_1^3a_2^4 \,.\, a_2^1a_1^2 \,.\, a_2^3a_3^4 \,.\, a_3^1a_2^2 \,.\, a_3^3a_1^4.$$

To sum up the results of the preceding work, the 16 heads with three systems of intransitivity give 41 groups with three systems of imprimitivity. The 7 heads with five systems of intransitivity give 42 groups with five systems of imprimitivity, but of these 13 groups contain also three systems of imprimitivity. Therefore there are 70 imprimitive groups of degree 15 as determined in this paper.

Primitive Substitution Groups of Degree Eighteen.

The main theorems employed in this investigation of primitive groups are the following, in which p is always to stand for a prime number.

I. *The order of a primitive group of degree n cannot exceed* $\dfrac{n!}{2.\,3\,\ldots\,p}$, *where* 2, 3, ... p *are the distinct primes which are less than* $\frac{2}{3}\,n$. (Burnside, Theory of Groups, p. 199).

II. *A group of degree* $p + x$ *or of degree* $2p + x$, $x > 2$, *cannot be more than* x *times transitive.* (Miller, Bull. A. M. S., v. IV, pp. 142, 143).

III. *If a primitive group of degree n contains a circular substitution of prime*

order p, the group is at least $(n-p+1)$-fold transitive. (Cole's tr. of Netto's Theory of Substitutions, p. 93).

IV. *A self-conjugate subgroup of a primitive group must be transitive.* (Burnside, l. c., p. 187).

V. *A self-conjugate subgroup of a $\varkappa$-ply transitive group of degree n $(2 < \varkappa < n)$ is in general at least $(\varkappa - 1)$-ply transitive. The only exception is that a triply transitive group of degree 2^m may have a self-conjugate subgroup of order 2^m.* (Burnside, l. c., p. 189).

VI. *A group G which is at least doubly transitive either must be simple or mus contain a simple group H as a self-conjugate subgroup. In the latter case no opera tion of G except identity is permutable with every operation of H. The only exceptions to this statement are that a triply transitive group of degree 2^m may have a self conjugate subgroup of order 2^m, and that a doubly transitive group of deyree p^m may have a self-conjugate subgroup of order p^m.* (Burnside, l. c., p. 192).

VII. *The substitutions of a transitive group G which leave a given symbol unchanged form a maximal subgroup G_1, which is one of a set of n conjugate subgroups, each leaving one of the n elements unaffected.* (Burnside, l. c., p. 140).

VIII. *The number of substitutions of degree $l < n$ contained in a transitive group of degree n is equal to the number of substitutions of this same degree l contained in the maximal subgroup G_1 of degree $n-1$ multiplied by $\frac{n}{n-l}$.* (Stated by Miller, Quar. Jour. of. Math. v. XXVIII, p. 215.)

IX. *The average number of elements in all the substitutions of a group is $n - \alpha$, n being the degree of the group and α the number of its transitive constituents.* (Frobenius, Crelle, t. CI, p. 287.)

X. Sylow's theorem, as stated by Burnside, l. c., p. 92, or by Sylow, "Théorèmes sur les groupes de substitutions," Math. Ann., v. V (1872), pp. 584 et seq.

XI. *The class of a primitive group of degree n is the same as the class of its maximal subgroup that leaves one element unaffected.*

While the preceding theorems are used throughout the work on primitive groups, the following are used mainly in the determination of simply transitive primitive groups.

XII. *A simply transitive primitive group G of degree n cannot contain a transitive subgroup of degree less than n.* (Miller, Quar. Jour. of Math., v. XXVIII, p. 215.

XIII. *When G_1 contains a self-conjugate subgroup H of degree $n - \alpha$, H must be intransitive, and it must be the transform with respect to substitutions of G of any one of $\alpha - 1$ other subgroups of G_1 $(H'_1, H'_2, \ldots. H'_{\alpha-1})$.* (Miller, Proc. Lon. Math. Soc., v. XXVIII, p. 534.)

XIV. *All the prime numbers which divide the order of one of the transitive constituents of G_1 divide also the orders of each of the other transitive constituents.*

Corollary I. *If one of the transitive constituents of G_1 is of a prime degree, each of its other transitive constituents is of the same or a larger degree, and the order of G_1 is the same as the order of the group formed by these other transitive constituents.*

Corollary II. *If the order of G_1 is not divisible by the square of a prime number, all its transitive constituents are of the same order, and G_1 is formed by establishing a simple isomorphism between them.* (Miller, l. c., p. 536.)

XV. *If a transitive constituent of G_1 is of a prime order, the order of G_1 is the same prime number, and G is of class $n - 1$.*

Corollary. *If G_1 contains a constituent of degree 2, its order is 2, and the degree of G is a prime number.* (Miller, l. c., p. 536.)

The above theorems are given in the form and with the symbols most convenient for use, and so are not always exact quotations from the papers and books referred to, while the references given are not always references to the original paper in which the theorem appeared.

Applying these theorems now to the special case in which $n = 18$, we proceed as follows:

Since $18 = 2.7 + 4$, by Theorem II a primitive group cannot be more than 4-ply transitive.

By Theorem I the order is seen not to exceed

$$\frac{18!}{2.3.5.7.11} = 2^{15}.\,3^{7}.\,5^{2}.\,7.\,13.\,17\,.$$

If the group included circular substitutions of orders 2, 3, 5, 7, 11, 13, it would be at least 17, 16, 14, 12, 8, 6-fold transitive respectively according to Theorem III. This is impossible; therefore circular substitutions of these orders are not present, and consequently we see at once that 11 and 13 cannot be factors of the order.

If the order includes the factor 7, then, by Theorem X, there is a subgroup of order 7. This must consist of the powers of a substitution composed of two cycles of 7 elements each, and it must be contained self-conjugately in a group of order $7 \, . \, 4m$ that interchanges transitively among themselves the four remaining elements. (Cf. Burnside, Theory of Groups, p. 202.) It is quite possible to establish a $(7\alpha, 1)$ isomorphism between an imprimitive group of degree 14 with the systems of imprimitivity 7, 7 and a transitive group of degree 4; therefore 7 may be a factor of the order.

A subgroup of order 5^2 cannot be present, as it would have to be intransitive with the systems of intransitivity 5, 5 or 5, 5, 5. In the one case, it would have to be contained self-conjugately in a group of order $5^2.\,8m$, in the other, in a group of order $5^2.\,3\,.\,m$. In either case, a circular substitution of order 5 would be present, which is impossible.

The factor 5 may be contained in the order, as it is possible to establish a (5, 1) isomorphism between the cyclical group of degree 15 and the cyclical group in the remaining three letters.

The order must, therefore, be a factor of $2^{15}.\,3^{7}.\,5\,.\,7\,.\,17$.

Simply Transitive Groups.

The maximal subgroup G_1 that leaves a_1 unaffected is intransitive (Theorem XII), and its order is, therefore, a factor of $2^{14}.\,3^{5}.\,5\,.\,7$. Moreover, its class cannot be less than 6, for if it were 2, 3 or 5, G_1 would necessarily contain a transitive subgroup of too low a degree, and it cannot be of class 4, if G is to be primitive. (Netto, l. c., p. 138.)

By Theorem XIV, Cor. I, it is evident that G_1 cannot contain a transitive constituent of degree 13 or 11; by Theorem XIV it cannot contain a transitive

constituent of degree 15 or 14, and by Theorem XV, Cor., it cannot contain a constituent of degree 2.

If one of the transitive constituents of G_1 is of degree 12, the other must be of degree 5. The isomorphism between the transitive groups of degrees 12 and 5 must be an $(\alpha, 1)$ isomorphism, where α itself may be equal to 1. By Theorem XIV, the order of the group of degree 5 must contain the factors 5, 3, 2; therefore this group must be either the alternating or the symmetric group of degree 5. If the isomorphism is more than simple, then the group of degree 12 must be an imprimitive group with 6 systems of imprimitivity. The head for such an imprimitive group as we require is the intransitive group of order 2 and degree 12 given by $[1,\ a_1a_2 \,.\, a_3a_4 \,.\, a_5a_6 \,.\, a_7a_8 \,.\, a_9a_{10} \,.\, a_{11}a_{12}]$. The group $(a_{13}a_{14}a_{15}a_{16}a_{17})$ pos contains

24 conjugate substitutions of order 5 and degree 5,
20 conjugate substitutions of order 3 and degree 3,
15 conjugate substitutions of order 2 and degree 4.

Corresponding to these in the group of degree 17, we have 1 substitution of degree 12 and order 2, 24 of degree $15 + 2\alpha$ and order 5 or 10, 24 others of degree $15 + 2\alpha'$ and order 5 or 10, 20 of degree $15 + 2\beta$ and order 3 or 6, ogether with 20 of degree $15 + 2\beta'$ and order 3 or 6, 15 of degree $12 + 2\gamma$ and order 2 or 4, and 15 of degree $12 + 2\gamma'$ and order 2 or 4, where

$$\alpha = 0,\ 1;\ \alpha' = 0,\ 1;\ \beta = 0,\ 1;\ \beta' = 0,\ 1;\ \gamma = 0,\ 1,\ 2;\ \gamma' = 0,\ 1,\ 2.$$

By Theorem IX, the following equation must be satisfied:

$$12 + 24\,(15 + 2\alpha) + 24\,(15 + 2\alpha') + 20\,(15 + 2\beta) + 20\,(15 + 2\beta') + 15\,(12 + 2\gamma) + 15\,(12 + 2\gamma') = 120 \,.\, 15.$$

The only type of solution is given by $\alpha = \beta = \gamma = \beta' = 0$, $\alpha' = 1$, $\gamma' = 2$. G_1, therefore, contains both a self-conjugate subgroup of degree 12 and order 2, and 15 conjugate subgroups of the same type. But, by Theorem XIII, only 5 such conjugate subgroups should exist if this group is to be the G_1 of a simply transitive primitive group. This intransitive group gives us, therefore, no such group as we require.

For precisely the same reason the intransitive group formed by establishing a (2, 1) isomorphism between an imprimitive group of degree 12 and order 240,

and the symmetric group of degree 5 cannot be employed in the formation of a simply transitive primitive group of degree 18.

If the isomorphism is simple, the group G_1, including the alternating group of degree 5, must contain 24 substitutions of degree 10 or 15 and order 5, 20 of degree 6, 9, 12 or 15 and order 3, 15 of degree 6, 8, 10, 12, 14 or 16 and order 2. The following equation must, therefore, be satisfied: $24(10+\alpha)+20(6+\beta)+15(6+\gamma)=60\times 15$, where $\alpha=0, 5$; $\beta=0, 3, 6, 9$; $\gamma=0, 2, 4, 6, 8, 10$. The only solution is $\alpha=5$, $\beta=9$, $\gamma=10$. G_1 is, therefore, of class 15.

A group G of degree 18, formed with the help of this G_1 and, therefore, of order $60 \,.\, 18$, contains 36 conjugate subgroups of order 5, each of which is contained self-conjugately in a group of order 30. As each of these subgroups of order 5 is already self-conjugate in a group of order 10, the construction of the generating substitutions of such a group is an easy matter. G_1 is generated by

$$s = a_1a_3a_7a_5a_9 \,.\, a_2a_4a_8a_6a_{10} \,.\, a_{13}a_{14}a_{15}a_{16}a_{17}$$

and by
$$t = a_1a_3 \,.\, a_2a_4 \,.\, a_5a_{12} \,.\, a_7a_8 \,.\, a_9a_{10} \,.\, a_6a_{11} \,.\, a_{13}a_{15} \,.\, a_{14}a_{17},$$

and contains, as one of the above-mentioned groups of order 10, the group generated by

$$u = st \quad = a_3a_8a_{11}a_6a_9 \,.\, a_4a_7a_{12}a_5a_{10} \,.\, a_{13}a_{17}a_{15}a_{16}a_{14},$$
$$v = s^{-1}ts = a_1a_2 \,.\, a_3a_7 \,.\, a_4a_8 \,.\, a_5a_6 \,.\, a_9a_{12} \,.\, a_{10}a_{11} \,.\, a_{13}a_{15} \,.\, a_{14}a_{16}.$$

The group $\{u, v\}$ is a subgroup of a group of degree 18 and order 30 formed by establishing a (5, 1) isomorphism between an imprimitive group of degree 15 and order 30 with u and its powers as head and the symmetric group in the three elements $a_1a_2a_{18}$. The question then reduces to that of the determination of a substitution of degree 18 and order 3 that will transform the head $\{u\}$ into itself, interchange cyclically the three systems of $\{u\}$, and be in its turn transformed into its square by v. An examination of all substitutions fulfilling these conditions results in finding none that do not give, when combined with other substitutions of G_1, substitutions that cannot possibly belong to a simply transitive primitive group containing G_1 as a maximal subgroup.

There is no primitive or imprimitive group of degree 12 simply isomorphic to the group $(abcdef)_{120}$; consequently, no isomorphism can be established between the symmetric group of degree 5 and a transitive group of degree 12.

There remains the question whether the symmetric group of degree 5 can be put in a simply isomorphic relation to one of the imprimitive groups of degree

12 and order 120 that have both six and two systems of imprivitivity. Such groups of degree 12, however, contain two self-conjugate subgroups of orders 2 and 60 respectively, and, therefore, are not in a simply isomorphic relation to the symmetric group of degree 5.

If one of the transitive constituents is of degree 10, the other can only be of degree 7. By Theorem XIV, the group of degree 10 must contain 7 as a factor of its order, and, therefore, must be either the alternating or the symmetric group. It is impossible to establish an isomorphic relation between either of these groups and one of degree 7 without introducing substitutions of too low a degree.

If one of the transitive constituents is of degree 9, the remaining constituent may be either intransitive in two systems of four elements each or intransitive in eight elements. The isomorphism can in neither case be simple, as an examination of all groups of degree 8 and orders equal to those of transitive groups of degree 9 shows that in each case a system of intransitivity of degree 2 enters, with the single exception of a group of order 144. Here, however, the group of degree 9 contains a substitution of order 8, while an inspection of the corresponding groups of degree 8 shows no substitution of that order.

The isomorphism is, therefore, an (α, β) isomorphism, where α and β are not simultaneously equal to one.

When neither α nor β is equal to one, G_1 must be formed from an imprimitive group of degree 9 and an intransitive group of degree 8. The order of each transitive constituent must contain 3 as a factor, and, therefore, the group of degree 8 must be some combination of the alternating and symmetric groups of degree 4 in two systems of elements. The only combinations possible, consistent with the requirements of class, are got by establishing a simple isomorphism between the two symmetric groups of degree 4 or between the two alternating groups of the same degree. Every relation of isomorphism established between these groups of degree 8 and any imprimitive groups of degree 9 consistent with the requirements of class, results in a G_1 that contains a self-conjugate subgroup of order 4 and degree 8, and no other subgroups of the same order. This case, therefore, gives no simply transitive primitive group.

When α becomes 1, the group of degree 8 must, as before, be composed of either the symmetric or the alternating groups of degree 4 in two sets of elements put into the relation of simple isomorphism. The order of such a group does not contain 9 as a factor; therefore this case gives no possible G_1.

When β becomes 1, the group of degree 9 must be imprimitive. No transitive group of degree 8 stands, however, in the given relation of isomorphism towards an imprimitive group of degree 9. The only permissible intransitive groups of degree 8 are combinations of the symmetric and alternating groups of degree 4 in two sets of elements, and none of these are isomorphic in the given way to any imprimitive group of degree 9.

If one of the transitive constituents is of degree 8, we may have the systems 8, 6, 3 or 8, 3, 3, 3. In both cases we have an $(\alpha, 1)$ isomorphism between an intransitive group of degree 14 and the symmetric group of degree 3. The group of degree 8 is not primitive, as no suitable isomorphic relation can be established between a primitive group of degree 8 and an imprimitive or an intransitive group of degree 6. The only imprimitive groups of degree 8 that can be used are those with the head $(1, a_1a_2 \, . \, a_3a_4 \, . \, a_5a_6 \, . \, a_7a_8)$ that are isomorphic to a group of degree 4 and order 12 or 24. Such groups, however, cannot be combined with the groups in the remaining 9 elements in such a way as to generate a group capable of being the G_1 of one of the required primitive groups.

The case in which G_1 contains a transitive constituent of degree 7 has already been discussed, as according to Theorem XIV, Cor. I, the remaining constituents must be of larger degree.

If G_1 contains a transitive constituent of degree 6, the systems may be either 6, 6, 5 or 6, 4, 4, 3. For the former system the only possible arrangement is to establish a simple isomorphism between the three groups $(a_1a_2a_3a_4a_5a_6)_{60}$, $(a_7a_8a_9a_{10}a_{11}a_{12})_{60}$, $(a_{13}a_{14}a_{15}a_{16}a_{17})$ pos, or between the groups $(a_1a_2a_3a_4a_5a_6)_{120}$, $(a_7a_8a_9a_{10}a_{11}a_{12})_{120}$, $(a_{13}a_{14}a_{15}a_{16}a_{17})$ all. An examination of the two groups G_1 formed from these isomorphisms shows that these are not the maximal subgroups of simply transitive primitive groups of degree 18.

If the systems are 6, 4, 4, 3, only the imprimitive groups of degree 6, the alternating and symmetric groups of degree 4, and the symmetric groups of degree 3 are involved. The group formed by the system 6, 4, 4 has an $(\alpha, 1)$ isomorphism to the group of degree 3, and this isomorphism cannot be simple. No combination of these groups can be found fulfilling all the necessary conditions.

The case in which G_1 contains a transitive system of degree 5 has already been discussed, as the remaining systems must be of degree greater than 5.

If G_1 contains a transitive system of degree 4, the only arrangement possible is 4, 4, 3, 3, 3. The groups involved are therefore the symmetric groups of degree 3 and 4, and the alternating group of degree 4. One group consistent with the

requirements of class is got by establishing a (1, 4) isomorphism between the group,

$$(a_9a_{10}a_{11}\,.\,a_{12}a_{13}a_{14}\,.\,a_{15}a_{16}a_{17})\text{ all, and }(a_1a_2a_3a_4\,.\,a_5a_6a_7a_8)\text{ all.}$$

This group contains, however, one and only one subgroup of degree 8 and order 2.

A second group is got by first establishing a (4, 1) isomorphism between $(a_1a_2a_3a_4.a_5a_6a_7a_8)$ all and $(a_9a_{10}a_{11})$ all; and then establishing a (12, 3) isomorphism between the group of order 24 so formed and the group $(a_{12}a_{13}a_{14}\,.\,a_{15}a_{16}a_{17})$ all. This group of degree 17 contains only one subgroup of order 4 and degree 8; therefore it cannot become a G_1.

G_1 cannot contain only systems of degree less than four, as in such a case a system of degree 2 would have to enter.

There is, therefore, no simply transitive primitive group of degree 18. This result when joined to all other determinations of similar groups shows that there is no simply transitive primitive group of degree $p+1$, p a prime number and ≤ 17.

Multiply transitive groups.

Among the transitive groups of degree 17 five contain a self-conjugate subgroup of order 17. These are of order 17, 2.17, 4.17, 8.17, 16.17 respectively, while all excepting the first are of class 16.

If a primitive group of degree 18 and order 18.17 existed, such a group would contain 18 conjugate subgroups of degree 17. It would therefore contain 17 substitutions of degree 18 and 18.16 of degree 17. By Sylow's theorem since $18.17 = 2.3^2.17$, such a group contains either 1 or 34 subgroups of order 3^2. A subgroup of this order must be intransitive, therefore cannot be self conjugate, and it is impossible to form 34 subgroups of order 9 from 17 substitutions of degree 18. No such group of degree 18 exists.

A primitive group of degree 18 and order $18.17.2 = 2^2.3^2.17$ would contain among its substitutions 153 of class 16 and order 2, 288 of class 17 and order 17, 170 of class 18. This group must contain either 1, 4, or 34 conjugate subgroups of order 3^2. As before, a subgroup of this order cannot be self-conjugate, as it is intransitive. If there were 4 conjugate subgroups, each would be self-conjugate in a group of order $3^2.17$ involving all 18 letters and necessarily transitive. Such a group is non-existent. If there were 34 conjugate subgroups they must be of degree 18, and there are not enough substitutions of class 18 to form all these subgroups.

A primitive group of degree 18 and order $18.17.4 = 2^3.3^2.17$ contains among its substitutions 476 of degree 18, 459 of degree 16, 288 of degree 17. According to Sylow's theorem it contains either 1, 3, 9, 17, 51, or 153 conjugate subgroups of order 2^3. Now the group leaving one element unchanged contains 17 conjugate subgroups of degree 16 and order 4; therefore the group of degree 18 contains 153 distinct conjugate subgroups of order 4; therefore it contains 153 conjugate subgroups of order 2^3. Each of these is contained self-conjugately in no larger group.

The number of systems of intransitivity in any one is got from the following equation, where x denotes the number of substitutions of degree 18 and α the number of systems:

$$18x + 16\,(7 - x) = 8\,(18 - \alpha), \text{ where } \alpha \neq 1,\ x < 8.$$

There are two sets of solutions, either $x = 0$, $\alpha = 4$, or $x = 4$, $\alpha = 3$.

The group of degree 17 is generated by,

$$s = a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}$$
$$t = a_2a_{14}a_{17}a_5 \cdot a_3a_{10}a_{16}a_9 \cdot a_4a_6a_{15}a_{13} \cdot a_7a_{11}a_{12}a_8\,,$$

where t and its powers form a self-conjugate subgroup of the group of order 2^3 and degree 18 that is now under discussion. It is impossible to so connect the systems and introduce the remaining elements that the first solution may give the group of order 2^3. Making use of the second solution we have only to combine with the group generated by t a substitution of degree 18 that connects the two remaining elements by a transposition, and unites the cycles of t in pairs. The 153 groups of order 2^3 give in this way $153.4 = 612$ distinct substitutions of degree 18, while there are only 476 in the group. This group of degree 18 does not exist.

If there is a primitive group of order $18\,.\,17\,.\,2^3 = 2^4.\,3^2.\,17$, it contains 288 substitutions of degree 17, 1071 of degree 16, 1088 of degree 18. The group of degree 17 which is generated by

$$s = a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}$$

and $$t = a_2a_{10}a_{14}a_{16}a_{17}a_9a_5a_3 \cdot a_4a_{11}a_6a_{12}a_{15}a_8a_{13}a_7\,,$$

contains 17 conjugate subgroups of degree 16 and order 8; therefore in the group of degree 18 there are 153 such conjugate subgroups, and each of these is self-conjugate in a group of order 2^4 and degree 18. Denoting by α the number of systems of intransitivity of this group of order 2^4, and letting x denote the number of substitutions of degree 18 contained in the group, we have the equation $18x + 16\,(15 - x) = 16\,(18 - \alpha)$, where $\alpha \neq 1$. There are two solutions, $x = 8$, $\alpha = 2$; $x = 0$, $\alpha = 3$. The first solution would involve a larger number of substi-

tutions of degree 18 than are actually present in the group under consideration. The second solution shows that the group must contain 153 conjugate subgroups of order 2^4 and degree 18 consisting of substitutions of class 16 only, and involving three systems of intransitivity. A substitution must therefore be combined with t that transforms t into one of its powers, and has its head in the group generated by t; moreover, this substitution must have as one of its cycles the transposition $(a_1 a_{18})$, and must have systems of intransitivity apart from this cycle consistent with the systems of t. Such a substitution is $\sigma = a_2a_{10} \cdot a_3 a_{14} \cdot a_9a_{17} \cdot a_5a_{16} \cdot a_6a_{13} \cdot a_7a_{11} \cdot a_8a_{12} \cdot a_1a_{18}$. The required group is therefore $\{s, t, \sigma\}$.

It is not necessary to prove that these three substitutions give a group of the required order, as such a group would be necessarily doubly transitive, and it is known that there is a doubly transitive group of degree 18 and of the required order. By the mode of construction of the substitutions, it is evident that there is only the one type of group of this degree and order.

Any primitive group of degree 18 and order $18 \cdot 17 \cdot 16$ contains 2312 substitutions of degree 18, 288 of degree 17, 2295 of degree 16. The group of degree 17 contains 17 conjugate cyclical subgroups of degree 16 and order 16, therefore, the group of degree 18 contains 153 subgroups of order 16, each of which is self-conjugate in one of 153 conjugate subgroups of order 32. Giving α and x the usual meanings, we find that the group of order 32 involves the equation $18x + 16\,(31 - x) = 32\,(18 - \alpha)$, where $\alpha \neq 1$. The only solution is $\alpha = 2$, $x = 8$; therefore, the group of degree 12 and order 32 must be intransitive with two systems of intransitivity, and must contain 8 substitutions of degree 18, 23 of degree 16. We have to add, therefore, to the cyclical group of degree 16, 8 substitutions of degree 16 and 8 of degree 18, all of them containing as one cycle the transposition of the remaining two letters.

The group of degree 17 and order $17 \cdot 16$ has as generators

$$s = a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}$$

and
$$u = a_2a_4a_{10}a_{11}a_{14}a_6a_{16}a_{12}a_{17}a_{15}a_9a_8a_5a_{13}a_3a_7.$$

The substitution $\tau = a_1a_{18} \cdot a_4a_7 \cdot a_3a_{10} \cdot a_5a_{14} \cdot a_6a_8 \cdot a_9a_{16} \cdot a_{11}a_{13} \cdot a_{12}a_{15}$ generates with s and u the required group of degree 18 and order $17 \cdot 16 \cdot 18$. A triply transitive group of such an order is known to exist (Burnside, l. c., p. 158); so no further proof that $\{s, u, \tau\}$ is a group is necessary. It is easy to see that the even substitutions of the group just found form the simple group of order $18 \cdot 17 \cdot 8$.

The three remaining transitive groups of degree 17 each contains 120 conjugate subgroups of order 17. They are of orders 15 . 16 . 17, 15 . 16 . 17 . 2, 15 . 16 . 17 . 4 respectively.

The group of degree 18 and order 15 . 16 . 17 . 18 would necessarily contain 816 conjugate subgroups of order 5. Each is self-conjugate in a group of order 90 connecting the remaining three elements transitively. This group is intransitive with two transitive constituents, one of degree 15 and order 90, the other of degree 3. The first, however, is non-existent, therefore, the group of degree 18 is non-existent.

The two remaining groups also, if they can generate primitive groups of degree 18, would generate groups that each contain 816 conjugate subgroups of order 5. In the one case, we should have to make use of an intransitive group containing as a transitive constituent a group of degree 15 and order 180, in the other, the transitive constituent would enter as a group of degree 15 and order 360. Both of these groups are non-existent; therefore, the three groups of degree 17, at present under discussion, furnish us with no new groups of degree 18.

As the case now stands, the conclusion arrived at may be summed up as follows:

There are no simply transitive primitive groups of degree 18, and in addition to the symmetric and alternating groups, there are only two multiply transitive groups of this degree, viz., the two given by

$$\left.\begin{cases} (a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}), \\ (a_2a_{10}a_{14}a_{16}a_{17}a_9a_5a_3 \,.\, a_4a_{11}a_6a_{12}a_{15}a_8a_{13}a_7), \\ (a_2a_{10} \,.\, a_3a_{14} \,.\, a_9a_{17} \,.\, a_5a_{16} \,.\, a_6a_{13} \,.\, a_7a_{11} \,.\, a_8a_{12} \,.\, a_1a_{18}), \end{cases}\right\} \text{of order } 2448,$$

$$\left.\begin{cases} (a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}), \\ (a_2a_4a_{10}a_{11}a_{14}a_6a_{16}a_{12}a_{17}a_{15}a_9a_8a_5a_{13}a_3a_7), \\ (a_1a_{18} \,.\, a_4a_7 \,.\, a_3a_{10} \,.\, a_5a_{14} \,.\, a_6a_8 \,.\, a_9a_{16} \,.\, a_{11}a_{13} \,.\, a_{12}a_{15}), \end{cases}\right\} \text{of order } 4896.$$

The second of these is triply transitive, and contains the first, which is doubly transitive and simple, as a self-conjugate subgroup.

The works consulted in the preparation of this paper have included, in addition to the standard works on the subject by Jordan, Serret, Netto, and Burnside, the following papers:

Askwith, "On Possible Groups of Substitutions that can be formed with 3, 4, 5, 6, 7 Letters Respectively." Quar. Jour. Math., v. XXIV (1890), pp. 111–167.

"On Groups of Substitutions that can be formed with Eight Letters." Quar. Jour. Math., v. XXIV (1890), pp. 263–331.

"On Groups of Substitutions that can be formed with Nine Letters." Quar. Jour. Math., v. XXVI (1892), pp. 79–128.

Cayley, "On Substitution Groups for Two, Three, Four, Five, Six, Seven, and Eight Letters." Quar. Jour. Math., v. XXV (1891), pp. 71–88, 137–155.

Cole, "List of the Substitution Groups of Nine Letters." Quar. Jour. Math., v. XXVI (1892), pp. 372–388.

"The Transitive Substitution Groups of Nine Letters." Bull. New York Math. Soc., v. II (1893), pp. 250–258.

"List of the Transitive Substitution Groups of Ten and of Eleven Letters." Quar. Jour. Math., v. XXVII (1894), pp. 39–50.

"Note on the Substitution Groups of Six, Seven, and Eight Letters." Bull. New York Math. Soc., v. II (1893), pp. 184–190.

Miller, "Intransitive Substitution Groups of Ten Letters." Quar. Jour. Math., v. XXVII (1894), pp. 99–118.

"List of Transitive Substitution Groups of Degree Twelve." Quar. Jour. Math., v. XXVIII (1896), pp. 193–231.

"Note on the Transitive Substitution Groups of Degree Twelve." Bull. Amer. Math. Soc., v. I (1894–1895), pp. 255–258.

"On the Transitive Substitution Groups of Degrees Thirteen and Fourteen." Quar. Jour. Math., v. XXIX (1898), pp. 224–249.

"On the Primitive Substitution Groups of Degree Fifteen." Proc. Lon. Math. Soc., v. XXVIII (1896–1897), pp. 533–544.

"On the Primitive Substitution Groups of Degree Sixteen." Amer. Jour. Math., v. XX (1898), pp. 229–241.

"On the Transitive Substitution Groups of Degree Seventeen." Quar. Jour. Math., v. XXXI (1899), pp. 49–57.

"A Simple Proof a Fundamental Theorem of Substitution Groups, and Several Applications of the Theorem." Bull. Amer. Math. Soc., v. II (1895–1896), pp. 75–77.

"On the Limit of Transitivity of the Multiply Transitive Substitution Groups that do not Contain the Alternating Group." Bull. Amer. Math. Soc., v. IV (1897–1898), pp. 140–143.

Jordan, "Sur la classification des groupes primitifs." Comptes Rendus, t. LXXIII (1871), pp. 853–857.

"Sur l'énumeration des groupes primitifs pour les dix-sept premiers degrés." Comptes Rendus, t. LXXV (1872), pp. 1754–1757.

PHILADELPHIA, *January* 1901.

ON A RECENT METHOD FOR DEALING WITH THE INTERSECTIONS OF PLANE CURVES*

BY

CHARLOTTE ANGAS SCOTT

Introduction.

1. In considering NOETHER'S theorem under its geometrical aspect, a question of interpretation forces itself into notice. In order that a curve $F = 0$ may have an equation of the form $Pu + Qv = 0$, where $u = 0$, $v = 0$ are given curves, the coefficients in the expression F must satisfy certain conditions; any one intersection of u, v being taken as origin, the conditions arising from this one affect the coefficients of terms whose degree does not exceed a certain value. Of these conditions some bear their interpretation on their face—the curve F must have a multiple point of a certain order, with tangents possibly given; but what is the geometric meaning of the others?

The determination of their precise algebraic construction is the first stage in the inquiry, and to this question an answer has been given in a very simple and significant form in a recent memoir by Dr. F. S. MACAULAY.† His explanation of the nature of the conditions is applicable not only to the so-called simple case, when the two curves u, v have no contact at any common point, but also to the general case, when either curve alone presents a singularity of any degree of complexity, and the two have contact of however elaborate a nature.

The conditions are simply the vanishing of (1) a single linear function of the coefficients, and (2) all functions obtained from it by a particular process of derivation. If we denote the coefficient of $x^r y^q$ by $z_{r,q}$, or, more conveniently, the coefficient of $x^{p-q} y^q$ by z^p_q, this process is simply the repeated and combined use of the two operators:

$D_x \equiv$ diminish every r by unity, $D_y \equiv$ diminish every q by unity,

or,

$D_x \equiv$ diminish every index p by unity, with the understanding that z^p_q is zero, for $p < q$,

$D_y \equiv$ diminish every index p and suffix q by unity.

* Presented to the Society December 27, 1901. Received for publication February 11, 1902.

† Proceedings of the London Mathematical Society, vol. 31 (1899), pp. 381–423; see also vol. 32 (1900), pp. 418–430.

The order in which the operators are used is obviously a matter of absolute indifference. Thus, for example, if the original condition is $E=0$, where

$$E = az_0^3 + bz_1^3 + cz_2^3 + dz_3^3 + ez_0^2 + fz_1^2 + gz_2^2 + hz_0^1 + kz_1^1 + lz_0^0,$$

then

$$D_x E = az_0^2 + bz_1^2 + cz_2^2 + ez_0^1 + fz_1^1 + hz_0^0 = 0\,,$$

$$D_y E = bz_0^2 + cz_1^2 + dz_2^2 + fz_0^1 + gz_1^1 + kz_0^0 = 0\,,$$

$$D_x^2 E = az_0^1 + bz_1^1 + ez_0^0 = 0\,,$$

$$D_x D_y E = bz_0^1 + cz_1^1 + fz_0^0 = 0\,,$$

$$D_y^2 E = cz_0^1 + dz_1^1 + gz_0^0 = 0\,,$$

$$D_x^3 E = az_0^0 = 0\,,\quad D_x^2 D_y E = bz_0^0 = 0\,,\quad D_x D_y^2 E = cz_0^0 = 0\,,\quad D_y^3 E = dz_0^0 = 0\,.$$

These four equations reduce to the one, $z_0^0 = 0$; the three above are equivalent to two only, namely, $z_0^1 = 0$, $z_1^1 = 0$, unless such relations hold among a, b, c, d as will reduce these three to one, namely, $a/b = b/c = c/d$. The two before these give

$$az_0^2 + bz_1^2 + cz_2^2 = 0,\quad bz_0^2 + cz_1^2 + dz_2^2 = 0\,,$$

that is

$$z_0^2 : z_1^2 : z_2^2 = bd - c^2 : bc - ad : ac - b^2;$$

and the original equation can be simplified by the omission of the terms now known to be zero. Thus it is seen that there is a double point with given tangents, the coefficients of the terms of the third degree being moreover subject to one linear condition. The number of independent equations in this system is six.

The set of equations, consisting of a single "prime equa ion" and all its derivates, Dr. Macaulay speaks of as a "one-set system"; the "base-point" thus defined (that is, the point together with the entire specified nature of the curve) he calls a "one-set point." If t prime equations are necessary for the complete specification, the point is a "t-set point."

2. The second of the two memoirs contains certain applications of the theory which is developed in the first memoir. The principal theorems there obtained, but in a different order, are:

(1) *the complete intersection of two curves u, v determines a one-set point* (pp. 394–400);

(2) *the complete intersection of $t+1$ curves $u_0, u_1, \cdots, u_t$ determines a t-set point* (pp. 419–423);

(3) *the number of points of intersection of two curves at one common point is the same as the number of equations contained in the one-set system afforded by the intersection* (pp. 388–393).

The whole development of the theory, as well as the proofs of these theorems, is elaborate and complicated; for instance, certain theorems of residuation are proved, and on these the proof of (2) is based. These theorems are, however, far removed from the inherent simplicity of the conception, and they throw no light on the very interesting character of the equations. On account of the intrinsic interest, and, I believe, importance of the central idea, it seems worth while attempting to present the theory with some fulness, but in a more direct manner. In this recasting of Dr. MACAULAY's material I have slightly inverted the definitions of his original memoir, and have given entirely different proofs of theorems (1) and (2). These proofs are in a different order of ideas; by means of a *theorem of ascent* I determine with precision the nature of the set of equations, after which examination the proofs of the two theorems follow immediately. The proof of (3) (Theorem I in the original memoir), is practically the same as the proof there given, but the preliminary investigation into the nature of the equations makes it somewhat more simple.*

3. Some preliminary remarks will perhaps make the trend of the argument clearer. As these are simply of a general and explanatory character, by no means essential to the formal treatment, some freedom is exercised in the use of certain phrases, which in a different context might challenge criticism. We begin by postulating elements of which the simplest kind is the usual directed linear infinitesimal element; these are combinations of infinitesimal arcs of all possible kinds, connected (as members of one complete branch) or entirely disconnected. Thus, for example, two arcs through the origin determine a double point; if no information is given as to the tangents, this element has two initial degrees of freedom; if the tangents are to be a pair in an assigned involution of line-pairs, the element has only one initial degree of freedom. We are concerned only with the degrees of freedom of the element assigned by the given conditions; we have nothing to do with what happens to the separate arcs afterwards. Such an element is spoken of as a *base-point*. If the element has one initial degree of freedom, the base-point is a one-set point; if it has t degrees of freedom, the point is t-set. Thus an unconditioned k-point (multiple point of order k) is necessarily a k-set point, for the element is composed of k simple arcs with unspecified tangents, and has, therefore, k degrees of freedom. But if any relation is to hold among the branches, this diminishes the number of degrees of freedom, and the point is a t-set point, where $t < k$. Any branch may be separately specified to any extent: but if then left free to wave independently of the others, the result still holds.

* The proof as given in the memoir, under its most natural interpretation (for it is extremely obscure) is open to criticism also on the score of rigor, but I have received from the author a modified form of some of the statements, designed to make it clear that when properly interpreted they are not open to this criticism. The obscurity however remains.

If the point be t-set, let $u_0, u_1, \cdots, u_t$ be curves, otherwise independent, on which it exists; then it exists also on

$$X_0 u_0 + X_1 u_1 + \cdots + X_t u_t = 0,$$

where $X_0, X_1, \cdots, X_t$ are general polynomials in x, y; and as this has initially t degrees of freedom at the origin the system may be expected to include all curves endowed with this point.

The equations proper to any base-point may be looked upon as specifying the nature of the curve in an infinitesimal region surrounding the point; when the curve enters this region, it comes under the influence of the prime equations, t in number if the point is t-set; but it is by no means necessary that the curve come under the influence of all t prime equations simultaneously.

I. *Nature and arrangement of the equations.*

4. Our object is to determine the *nature of the conditions to which the coefficients in the equation of a curve,* $u = 0$, *must be subject in order that the equation may be expressible in the form* $X_0 u_0 + X_1 u_1 = 0$, or, we may say, in order that the curve may be a member of the system $X_0 u_0 + X_1 u_1 = 0$, where u_0, u_1 are given curves, while X_0, X_1 are entirely arbitrary polynomials in x, y. We consider this question only as regards the conditions due to the nature of the curves at the origin, this having been taken at a point common to u_0, u_1. All curves of the system have something in common at the origin, even if it be only, as in the simplest case, that they pass through it. Similarly, as regards the more general system of the same type,

$$X_0 u_0 + X_1 u_1 + X_2 u_2 + \cdots = 0,$$

which obviously includes the general linear system

$$h_0 u_0 + h_1 u_1 + h_2 u_2 + \cdots = 0,$$

there arises the question of the behavior of the curves at the origin; in other words, the determination of their common characteristics at that point.

The form of the conditions appear at once from this delimitation of the field of investigation. (1) If X be any polynomial, the reducible curve $Xu = 0$ has the same branches at the origin as $u = 0$, with others in addition if the curve $X = 0$ itself passes through this point. Hence the coefficients in Xu satisfy the equations to which the coefficients of u are subject. Since X may be taken to be a mere numerical multiplier, this proves that every equation is homogeneous in the coefficients of u. (2) Inasmuch as any curve $u + kv = 0$ belongs to the system if u, v are members, it follows that the coefficients in $u + kv$ satisfy the equations for all values of k. Consequently the coefficients of u enter only in the first degree; that is, every equation with which we are concerned is linear

in the coefficients of u. (3) If for the polynomial X we take simply $x^h y^k$, it is evident that any equation satisfied by the coefficients of u in virtue of the base-point is satisfied also by the corresponding coefficients of $x^h y^k u$.* Thus if $\sum \lambda_{r,q} z_{r,q} = 0$, and if we form this same expression for $x^h y^k u$, the term $x^r y^q$ has now the coefficient $z_{r-h,\,q-k}$, hence we have $\sum \lambda_{r,q} z_{r-h,\,q-k} = 0$. The original equation being denoted by $E = 0$, the values 1, 0 for h, k give $D_x E = 0$, and the values 0, 1 give $D_y E = 0$, where D_x, D_y have the meanings explained in the introduction. The equation derived from $E = 0$, as above, is $D_x^h D_y^k E = 0$. In the index and suffix notation, the result obtained is that if $\sum \lambda_q^p z_q^p = 0$, then for any set of values l, m, such that $p - l \geqq q - m$, $l \geqq m$, $\sum \lambda_q^p z_{q-m}^{p-l} = 0$. If this be denoted by E_m^l, then $D_x E$ is E_0^1, $D_y E$ is E_1^1. The equation $E = 0$ is looked upon as the *prime equation* or source; the others are derivates or descendants. By the *degree* of an equation is to be understood the highest index p that occurs. If an equation is of degree p, then any pth derivate is simply $z_0^0 = 0$.

5. One prime equation with all its derivates may not account for all the equations to which the coefficients are subject. If t prime equations are necessary, the point is said to be *t-set.* For instance, the two prime equations $z_0^1 = 0$, $z_1^1 = 0$, with the one derivate $z_0^0 = 0$, determine a two-set point, a simple node. If all the equations are accounted for by the one prime equation and its derivates, the point is *one-set.*

An equation that is prime for one base-point may present itself as a derivate for a more extensive base-point. For example, on the curve

$$x + y + x^2 + 3xy + 4y^2 + x^3 + 2x^4 + y^4 + \cdots = 0,$$

the curve

$$x + y + 2x^2 + 4x^3 + y^3 + 6x^4 + \cdots = 0$$

determines a one-set point for which the prime equation is $z_0^2 - z_1^2 + z_2^2 - 2z_0^1 = 0$; while the curve

$$x + y + x^2 + y^2 - 7y^3 + 5x^4 + 2y^4 + \cdots = 0$$

determines a one-set point with the prime equation

$$z_0^3 - z_1^3 + z_2^3 - z_3^3 - 2z_0^2 + 2z_2^2 - 7z_0^1 = 0,$$

whose x-derivate is $z_0^2 - z_1^2 + z_2^2 - 2z_0^1 = 0$.

---

* Although this brings in other branches at O, the existing branches are not affected. The wording in the text can, however, be varied; $(1 + bx^h y^k)u$ has exactly the same base-point as u; hence, if $\sum \lambda_{r,q} z_{r,q} = 0$, it follows that $\sum \lambda_{r,q}(z_{r,q} + bz_{r-h,\,q-k}) = 0$, from which, by subtraction, the result follows as in the text.

6. *Law of unit increase.*—From the mode of formation of the derivates, it is seen that the number of any degree may increase by unity for every diminution in the degree; we have, in fact, the scheme

$$\begin{array}{ccccccl} & & E & & & \ldots\ldots & \text{of degree } p\,, \\ & \diagup & & \diagdown & & & \\ & E_0^1 & & E_1^1 & & \ldots & \text{of degree } p-1\,, \\ \diagup & & \diagdown\ \diagup & & \diagdown & & \\ E_0^2 & & E_1^2 & & E_2^2 & \ldots & \text{of degree } p-2\,, \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\ldots, \end{array}$$

where the two derivates of any equation are placed obliquely below it, the x-derivate to the left, the y-derivate to the right. Thus E_1^2 is the x-derivate of E_1^1 and the y-derivate of E_0^1. There may however be agreement among the derivates; in the example of the last paragraph, the y-derivate of

$$z_0^3 - z_1^3 + z_2^3 - z_3^3 - 2z_0^2 + 2z_2^2 - 7z_0^1 = 0$$

is $-z_0^2 + z_1^2 - z_2^2 + 2z_1^1 = 0$, which is the same as the x-derivate, in virtue of the relation, shown by the next derivates, $z_0^1 - z_1^1 = 0$. On account of this possibility, all that can be asserted at present is that in any one-set system the increase, for unit decrease in degree, cannot be more than unity. This warrants no conclusions as to the total number of equations of any degree for a t-set point, inasmuch as there may be prime equations of that degree.

7. *Whether the point be one-set or t-set, if the greatest number of equations of any one degree be k (where obviously $k \geqq t$), the point is multiple of order k.* By hypothesis, there are k different equations of some degree p; that is, from these there can be formed linearly no equation of lower degree. It is to be proved that the number of independent equations of any lower degree p' cannot be less than k, provided that $p' \geqq k$. As every equation gives at least one representative in the next lower degree, it has to be shown that these can be chosen so that no two agree as to their highest terms; or, more generally, so that the highest terms cannot be eliminated from any number of the equations. The choice to be made is that between x-derivates and y-derivates.

If some of the x-derivates can be combined in such a manner as to eliminate the highest terms, then their sources can be similarly combined so as to eliminate the highest terms with the exception of z_p^p; for any term $\lambda_q^p z_q^{p-1}$ in a derivate arises from a term $\lambda_q^p z_q^p$ in the source, hence all terms in the source, except z_p^p, are represented in the derivates. This combination of the sources yields an equation by which z_p^p is given in terms of z's with a lower index. There cannot be two such equations, since by hypothesis it is not possible to eliminate every z^p.

Hence in forming the x-derivates one of two things must happen: either (1) the k independent equations of degree p give, by means of their x-derivates, k inde-

pendent equations of degree $p-1$, which is the desired result; or (2) the k equations of degree p can be written as one, giving z_p^p in terms of lower z's, and $k-1$ that do not contain z_p^p. The x-derivates of these give $k-1$ equations, and the y-derivate of the first gives z_{p-1}^{p-1} in terms of lower z's. If this is independent of the $k-1$, the result follows; if it is not, these $k-1$ equations can be combined so as to eliminate every z^{p-1} except z_{p-1}^{p-1}, and the similar combination of their sources will not contain any z^p except z_{p-1}^p; this combination is therefore an equation giving z_{p-1}^p in terms of lower z's. The k equations of degree p are now

(1) one, giving z_p^p in terms of lower z's,

(2) one, giving z_{p-1}^p in terms of lower z's,

(3) $k-2$, not containing z_p^p, z_{p-1}^p.

At every repetition of this argument, either the immediately favorable case presents itself, or we continue with the apparently unfavorable case; this goes on until the k equations of degree p are all arranged so as to give, separately, $z_p^p, z_{p-1}^p, z_{p-2}^p, \cdots, z_{p-k+1}^p$ in terms of lower z's. The y-derivates of these, which are all different, yield k independent equations of degree $p-1$.

This argument holds so long as $p-k+1 > 0$, that is, down to and including the case $p' = k$. Thus we have k equations of the type

$$(z_0^k, z_1^k, \cdots z_i^k)^1 = \text{a linear function of lower } z\text{'s}.$$

(i) If every z^k is present, these k equations can be solved so as to give every $z_q^k (q \neq k)$ in terms of z_k^k and lower z's. The x-derivates then give k independent equations

$$z_0^{k-1} = \text{a linear function of lower } z\text{'s},$$

$$z_1^{k-1} = \text{a linear function of lower } z\text{'s},$$

$$\vdots$$

$$z_{k-1}^{k-1} = \text{a linear function of lower } z\text{'s}.$$

Proceeding with the x-derivates, we obtain finally, in reverse order,

$$z_0^0 = 0,$$

$$z_0^1 = [az_0^0 =]\,0, \quad z_1^1 = 0,$$

$$z_0^2 = (z_0^1, z_1^1, z_0^0)^1 = 0, \quad z_1^2 = 0, \quad z_2^2 = 0,$$

$$\cdots\cdots\cdots\cdots$$

$$z_0^{k-1} = 0, \quad z_1^{k-1} = 0, \quad z_2^{k-1} = 0, \quad \cdots, \quad z_{k-1}^{k-1} = 0.$$

These values for the lower z's reduce the k equations of degree k to the form

$$z_0^k : z_1^k : z_2^k : \cdots : z_k^k = \text{known values}.$$

Thus the point is multiple, of order k, with determinate tangents.

(ii) If any one z^k be absent, the remaining z^k's, k in number, are obtainable linearly in terms of lower z's. The x-derivates of those before the missing one, and the y-derivates of those after, give the k independent equations exactly as in the preceding case. We still have the multiple point of order k; and moreover every z^k, except the missing one, is zero. If this unmentioned one be z_l^k, the tangents are $x^{k-l}y^l = 0$. If now $p < k$, this amounts to saying that we have k independent equations of degree $\leqq k-1$. As the number of z's of any degree h is equal to $h+1$, the lowest possible value for this degree is $k-1$; the equations in this case can be written so as to give $z_0^{k-1}, z_1^{k-1}, z_2^{k-1}, \cdots, z_{k-1}^{k-1}$ linearly in terms of lower z's, and exactly as before, by means of x-derivates it is seen that every z up to and including every z^{k-1} must vanish. In this case however the tangents are not necessarily determined.

The general conclusion is therefore that if for any degree p there are as many as k independent equations, this being the greatest number for any degree, then the point is multiple of order k, with tangents which are determinate, if $p \geqq k$; possibly conditioned in some manner, if $p = k-1$; entirely unconditioned, if $p = k-1$ and the equations are prime.

8. Any equation of the set may of course be modified by the addition of multiples of any other of the equations. When it is found that the point is multiple of order k, so that every z^p $(p < k)$ is zero, all these lower z's are to be struck out of the equations. Another possible simplification can sometimes be detected. If a linear function of an expression E and some of its derivates presents itself as an equation of the set, this can be replaced by $E = 0$. For taking all the derivates down to and including the pth derivates, where p is the degree of E, we obtain

$$E_0^p = 0,\ E_1^p = 0,\ \cdots,\ E_p^p = 0;\ E_0^{p-1} = \text{linear functions of } E^p\text{'s} = 0,$$

and so on, till finally $E = 0$.

9. The argument by which it was shown that the point is multiple of order k depends on proving that the number of equations cannot diminish as the degree diminishes, so long as this degree $\geqq k$. It has been remarked that in general the number of equations derivable from any one prime equation increases by unity when the degree is diminished by unity. The prime equation $E = 0$, of degree p, yields two derivates E_0^1 and E_1^1, and consequently two equations of degree $p-1$; thus there is at least a double point, unless these two derivates are absolutely equivalent. To exhibit them as equivalent, it may be necessary

to modify them by the addition of multiples of lower derivates, but this will not affect the terms of highest degree. Let the prime equation be

$$E = a_0 z_0^p + a_1 z_1^p + \cdots + a_p z_p^p + (z)^{p-1} + \cdots = 0\,;$$

the derived equations are

$$E_0^1 = a_0 z_0^{p-1} + a_1 z_1^{p-1} + \cdots + a_{p-1} z_{p-1}^{p-1} + (z)^{p-2} + \cdots = 0\,,$$

$$E_1^1 = a_1 z_0^{p-1} + a_2 z_1^{p-1} + \cdots + a_p z_{p-1}^{p-1} + (z)^{p-2} + \cdots = 0\,.$$

These agree as to their highest terms if, and only if, $a_0/a_1 = a_1/a_2 = \cdots = a_{p-1}/a_p$; that is, if the coefficients of the highest terms form a geometric progression. The prime equation can then be written

$$z_0^p + \mu z_1^p + \mu^2 z_2^p + \cdots + \mu^p z_p^p + (z)^{p-1} + \cdots = 0\,,$$

so that it is at once obvious that all derivates of any one degree are equivalent as regards their highest terms.

The equation $\mu E_0^1 - E_1^1 = 0$ is of degree lower than $p-1$; if it is expressible in terms of derivates of E_0^1, the two equations E_0^1 and E_1^1 are absolutely equivalent; from $E = 0$ we obtain by a single derivation only the one equation $E_0^1 = 0$. Similarly the three derivates of degree $p-2$ reduce to one only; for they are $D_x E_0^1$, $D_x E_1^1 (= D_y E_0^1)$, $D_y E_1^1$. Now $E_1^1 \equiv E_0^1$, therefore $D_y E_1^1 \equiv D_y E_0^1 \equiv D_x E_1^1 \equiv D_x E_0^1$; also $D_x E_1^1 \equiv D_x E_0^1$, thus all are equivalent to E_0^2. In like manner the next derivates reduce to one only, and so on. Hence unless the first two derivates are independent, there is but one derivate of any degree, and the point is not multiple. *If the point determined by a given prime equation is multiple, this fact will make itself felt at the first derivation.*

If now the equation $\mu E_0^1 - E_1^1 = 0$, of degree $< p-1$, is not expressible in terms of lower derivates of E_0^1, the two equations E_0^1 and E_1^1 are not absolutely equivalent; from $E = 0$ we obtain by a single derivation the equation $E_0^1 = 0$ and an equation of lower degree. Similarly at any stage in the derivation it may be possible to eliminate from the $m+1$ equations of degree $p'-1$ all the highest terms, thus obtaining an equation of degree $p''(< p'-1)$, to be substituted for one of the $m+1$ equations. If this new equation is expressible in terms of the derivates of the m others, it adds nothing to our knowledge; from the m equations of degree p' we obtain only m equations of degree $p'-1$. But if this new equation is not so expressible, it has to be taken into account when we arrive at degree p''.

10. In forming the scheme of equations, whether it be regular or interrupted, the identity of $D_x E_k^l$ and $D_y E_{k-1}^l$ makes it unnecessary to write down both x- and y-derivates of all the equations at any stage; it is enough to write down

the x-derivates of all, and the y-derivates of the last one (the pure y-derivate E^l_l). The simplest process is perhaps to form the y-derivates, writing these in an oblique line downwards to the right, and then write down a vertical line of x-derivates, starting from each of these. The simple or regular scheme, that in which no elimination of the highest terms from the equations of any one degree is possible, is then

$$\begin{array}{lllll l}
E & . & . & . & . \quad . \quad . \quad . & \text{of degree } p\,, \\
E^1_0 & E^1_1 & & & . \quad . \quad . \quad . & \text{of degree } p-1\,, \\
E^2_0 & E^2_1 & E^2_2 & & . \quad . \quad . & \text{of degree } p-2\,, \\
E^3_0 & E^3_1 & E^3_2 & E^3_3 & . & \text{of degree } p-3\,, \\
. \quad . \quad . & \text{etc.} & . \quad . \quad . & . \quad . & \text{etc.} & . \quad . \quad .
\end{array}$$

If now the arrangement is interrupted by the possibility of eliminating the terms of degree $p'-1$ from the $m+1$ equations of degree $p'-1$, so obtaining an equation $E'=0$ of degree $p''(\lt p'-1)$, for the derivates at this stage we can substitute $E_0, E_1, E_2, \cdots, E_{m-1}, E'_m$. As before, it suffices to take the x-derivates of $E_0, E_1, \cdots, E_{m-1}$, if both x- and y-derivates be taken of E'_m. Thus for degrees $p'-1$ to $p''+1$ included, there are m equations; at degree p'' there are $m+1$ equations, since E'_m is now to be taken into account, and the law of unit increase is resumed until again interrupted. It is convenient to speak of the equations of which x-derivates only need be taken as stationary, the other being progressive. As an illustration of such an interrupted scheme take the prime equation

$$\begin{aligned}E = z^8_0 - z^8_1 - z^8_3 - z^8_4 - 2z^8_5 - 3z^8_6 - 5z^8_7 - 8z^8_8 + 2z^7_0 + z^7_1 + z^7_2 + 3z^7_3 \\ + z^7_4 - z^7_5 + z^6_0 + 3z^6_1 + z^6_2 + 4z^6_3 + z^6_4 + 3z^5_0 + 2z^5_1 + z^5_2 + 8z^4_0 = 0\,.\end{aligned}$$

Here $E^2_0 + E^2_1 - E^2_2$ is of degree 5; there are only two independent equations of degree 6. The three of degree 5 are

$$E^3_0 = z^5_0 - z^5_1 - z^5_3 - z^5_4 - 2z^5_5 + 2z^4_0 + z^4_1 + z^4_2 + 3z^4_3 + z^4_4 = 0\,,$$

$$E^3_1 = -z^5_0 - z^5_2 - z^5_3 - 2z^5_4 - 3z^5_5 + z^4_0 + z^4_1 + 3z^4_2 + z^4_3 - z^4_4 = 0\,,$$

and

$$E^2_0 + E^2_1 - E^2_2 = E = 2z^5_0 - z^5_1 + 3z^5_2 + 5z^5_3 - z^5_5 + 3z^4_0 + 4z^4_2 + 5z^4_3 + z^4_4 = 0\,;$$

and since there are four of degree 4, namely

$$E^4_0 = z^4_0 - z^4_1 - z^4_3 - z^4_4 = 0\,,$$

$$E^4_1 = -z^4_0 - z^4_1 - z^4_2 - 2z^4_4 = 0\,,$$

$$D_x E = 2z^4_0 - z^4_1 + 3z^4_2 + 5z^4_3 = 0\,,$$

$$D_y E = -z^4_0 + 3z^4_1 + 5z^4_2 - z^4_4 = 0\,,$$

there is a 4-point, with determinate tangents, given by

$$z_0^4 : z_1^4 : z_2^4 : z_3^4 : z_4^4 = 82 : 94 : -45 : 13 : -25,$$

that is,

$$82x^4 + 94x^3y - 45x^2y^2 + 13xy^3 - 25y^4 = 0.$$

The diagrammatic representation of this set of equations is

$$\begin{array}{lllll l}
E & . & . & . & . & \text{of degree } 8, \\
E_0^1 & E_1^1 & . & . & . & \text{of degree } 7, \\
E_0^2 & E_1^2 & . & . & . & \text{of degree } 6, \\
E_0^3 & E_1^3 & E & . & . & \text{of degree } 5, \\
E_0^4 & E_1^4 & E_0 & E_1 & . & \text{of degree } 4.
\end{array}$$

In the regular scheme, if the prime equation be of degree p, the number of equations of any degree $k-1$ is $p+2-k$. So long as $k \geqq p+1-k$, the derivation can go on; but since a set of k equations of degree $k-1$ indicates a k-point, the value of p is determined by the equality $k = p+2-k$, and consequently $p = 2(k-1)$. If p has a greater value than this, the scheme is not regular.

11. It has been shown that if the first two derivates are absolutely equivalent, then there is only one derivate of any particular degree. The general theorem, of which this is a special case, is the following:

If the k derivates of degree p give rise to only k derivates, then the number remains stationary, and the point is consequently a k-point.

Let the k derivates of degree p be denoted by $E_0, E_1, E_2, \cdots, E_{k-1}$. If the set were regular, we should have at the next stage $k+1$ derivates, $E_0^1, E_1^1, E_2^1, E_{k-1}^1, E_k^1$, where $E_0^1 = D_x E_0$, etc., and $E_k^1 = D_y E_{k-1}$; but by hypothesis, these are equivalent to k only. There is therefore one linear relation connecting some or all of the E^1's; let the *last* E^1 involved in this be E_h^1, so that the relation can be written

$$E_h^1 = a_0 E_0^1 + a_1 E_1^1 + a_2 E_2^1 + \cdots + a_{h-1} E_{h-1}^1.$$

The independent derivates of this rank are now

$$E_0^1,\ E_1^1,\ \cdots,\ E_{h-1}^1;\quad E_{h+1}^1,\ \cdots,\ E_k^1.$$

In the next rank we have to take account only of the x-derivates of E_0^1 to E_{h-1}^1, the y-derivates of E_{h+1}^1 to E_k^1. For the only y-derivate not included among these is $D_y E_{h-1}^1$, which is the same as $D_x E_h^1$, and is therefore equal to $D_x(a_0 E_0^1 + a_1 E_1^1 + \cdots + a_{h-1} E_{h-1}^1)$; that is, it is a linear function of those

x-derivates that have been taken into account. Similarly, the only x-derivate apparently neglected is

$$\begin{aligned} D_x E^1_{h+1} &\equiv D_y E^1_h \\ &\equiv D_y(a_0 E^1_0 + a_1 E^1_1 + \cdots + a_{h-1} E^1_{h-1}) \\ &\equiv D_x(a_0 E^1_1 + a_1 E^1_2 + \cdots + a_{h-1} E^1_h). \end{aligned}$$

Now the x-derivates of $E^1_1, E^1_2, \cdots, E^1_{h-1}$ have been explicitly taken into account; and it has just been shown that $D_x E^1_h$ depends on these. Consequently no derivate has been neglected; all the derivates of the next rank are obtained by means of these h x-derivates and $k-h$ y-derivates; their number is therefore k. A precisely similar proof applies to the equations of the next rank, and so on. Thus if all the equations become stationary, they remain stationary. The diagram now presents such an appearance as the following, for which the system of equations is that proceeding from the prime equation

$$E = z^8_0 + 2z^8_1 - 4z^8_2 + z^8_3 + z^8_4 + 2z^8_5 - 4z^8_6 + z^8_7 + z^8_8 + z^7_0 - z^7_3 + z^7_4 - z^7_7 \\ + z^6_0 - z^6_1 + z^6_4 - z^6_5 + (z)^5 + (z)^4 + (z)^3 = 0\,.$$

$$\begin{array}{llll} E & . \quad . \quad . & . \quad . & \text{of degree } 8, \\ E^1_0 & E^1_1 & . \quad . & \text{of degree } 7, \\ E^2_0 & E^2_1 & E^2_2 \quad . & \text{of degree } 6, \\ E^3_0 & E^3_1 & E^3_2 \quad . & \text{of degree } 5, \\ E^4_0 & E^4_1 & E^4_2 \quad . & \text{of degree } 4, \\ E^5_0 & E^5_1 & E^5_2 \quad . & \text{of degree } 3, \\ E^6_0 & E^6_1 & E^6_2 \quad . & \text{of degree } 2. \end{array}$$

II. *The theorem of ascent.*

12. It is a simple matter to write down, beginning with the lowest terms, the general equation of a curve for which a given system of equations is satisfied. The converse operation, that of determining the equations satisfied by the coefficients of given curves that pass through the origin, though possibly lengthy, is simple enough, theoretically. As regards the coefficients of the lower terms, the equations can be found by a direct process: but for the higher terms, the process of ascent is more satisfactory. This is directly derived from a theorem now to be proved, but before entering upon this, it will be shown that *two curves u, v with a k-point at the origin satisfy either $k-1$ or k equations of degree k, and $k-2$, $k-1$, or k equations of degree $k+1$.*

Since every z with index $< k$ is zero, the equations of degree k are of the type

$$\lambda_0 z_0^k + \lambda_1 z_1^k + \cdots + \lambda_k z_k^k = 0 .$$

Let the given curves be

$$u = a_0^k x^k + a_1^k x^{k-1} y + \cdots + u_{k+1} + \cdots = 0 ,$$

$$v = b_0^k x^k + b_1^k x^{k-1} y + \cdots + v_{k+1} + \cdots = 0 ,$$

then the λ's are subject to the conditions

$$\lambda_0 a_0^k + \lambda_1 a_1^k + \cdots + \lambda_k a_k^k = 0 ,$$

$$\lambda_0 b_0^k + \lambda_1 b_1^k + \cdots + \lambda_k b_k^k = 0 .$$

Hence the general equation of degree k is

$$\begin{vmatrix} z_0^k & z_1^k & \lambda_2 z_2^k + \cdots + \lambda_k z_k^k \\ a_0^k & a_1^k & \lambda_2 a_2^k + \cdots + \lambda_k a_k^k \\ b_0^k & b_1^k & \lambda_2 b_2^k + \cdots + \lambda_k b_k^k \end{vmatrix} = 0 ,$$

where the λ's are arbitrary. There are therefore $k - 1$ equations of degree k, obtained by equating to zero the coefficients of the different λ's; these equations are given by any $k - 1$ independent determinants of the set

$$\left\| \begin{matrix} z_0^k & z_1^k & z_2^k & \cdots & z_k^k \\ a_0^k & a_1^k & a_2^k & \cdots & a_k^k \\ b_0^k & b_1^k & b_2^k & \cdots & b_k^k \end{matrix} \right\| .$$

If however the tangents to u, v at the origin are the same, so that

$$a_0^k : a_1^k : a_2^k : \cdots = b_0^k : b_1^k : b_2^k : \cdots ,$$

the λ's are subject to only one condition, namely,

$$\lambda_0 a_0^k + \lambda_1 a_1^k + \cdots + \lambda_k a_k^k = 0 ,$$

and the equations are those given by

$$\left\| \begin{matrix} z_0^k & z_1^k & \cdots & z_k^k \\ a_0^k & a_1^k & \cdots & a_k^k \end{matrix} \right\| = 0 ,$$

that is,

$$z_0^k : z_1^k : z_2^k : \cdots = a_0^k : a_1^k : a_2^k : \cdots ,$$

k equations which express simply that the tangents are given.

If some of the b's are proportional to some of the a's, the equations remain $k - 1$ in number. Every determinant can be written so as to contain one

column in which the b is not this same multiple of the a. The determinants are then of the two types

$$\begin{vmatrix} z_0^k & z_3^k & z_4^k \\ a_0^k & a_3^k & a_4^k \\ b_0^k & b_3^k & b_4^k \end{vmatrix} = 0, \qquad \begin{vmatrix} z_0^k & z_1^k & z_2^k \\ a_0^k & a_1^k & a_2^k \\ b_0^k & \mu a_1^k & \mu a_2^k \end{vmatrix} = 0,$$

of which the second reduces to $z_1^k : a_1^k = z_2^k : a_2^k$. Thus the case is in no way exceptional.

No values for the a's and b's can make the number of equations exceed k, inasmuch as $k+1$ equations of degree k would give for every z^k the value zero, thus determining the multiple point on the curve as of order $k+1$. Hence the number of equations of degree k is $k-1$ in the general case, k if the curves have all their tangents in common.

It is now to be determined how many equations there are of degree $k+1$. If there are $k-1$ equations of degree k, by means of these all z^k's can be expressed in terms of two only; if these are z_0^k and z_k^k, then any equation of degree $k+1$ is

$$\mu_0 z_0^{k+1} + \mu_1 z_1^{k+1} + \cdots + \mu_{k+1} z_{k+1}^{k+1} + \alpha z_0^k + \beta z_k^k = 0,$$

where the μ's are subject to the conditions

$$\mu_0 a_0^{k+1} + \mu_1 a_1^{k+1} + \cdots + \mu_{k+1} a_{k+1}^{k+1} + \alpha a_0^k + \beta a_k^k = 0,$$

$$\mu_0 b_0^{k+1} + \mu_1 b_1^{k+1} + \cdots + \mu_{k+1} b_{k+1}^{k+1} + \alpha b_0^k + \beta b_k^k = 0.$$

The x- and y-derivates of these give other equations to be satisfied, namely

$$\mu_0 z_0^k + \mu_1 z_1^k + \cdots + \mu_k z_k^k = 0,$$

$$\mu_1 z_0^k + \mu_2 z_1^k + \cdots + \mu_{k+1} z_k^k = 0;$$

hence the μ's must satisfy

$$\begin{aligned} \mu_0 a_0^k + \mu_1 a_1^k + \cdots + \mu_k a_k^k \qquad\qquad &= 0, \\ \mu_1 a_0^k + \cdots + \mu_k a_{k-1}^k + \mu_{k+1} a_k^k &= 0, \\ \mu_0 b_0^k + \mu_1 b_1^k + \cdots + \mu_k b_k^k \qquad\qquad &= 0, \\ \mu_1 b_0^k + \cdots + \mu_k b_{k-1}^k + \mu_{k+1} b_k^k &= 0. \end{aligned}$$

Thus the $k+4$ parameters ($k+2$ μ's, α, β) are subject to six equations, and so $k-2$ parameters are arbitrary. The number of equations of degree $k+1$ is therefore $k-2$ in general, though a linear relation connecting the six equations may increase this number. As however it is known that it cannot exceed the number of equations of degree k, it can only be $k-2$ or $k-1$.

If however there are k equations of degree k, every z^k is known in terms of any one, e. g., z_0^k. Thus the general equation of degree $k+1$ is now

$$\mu_0 z_0^{k+1} + \mu_1 z_1^{k+1} + \cdots + \mu_{k+1} z_{k+1}^{k+1} + a z_0^k = 0,$$

involving only $k+3$ parameters. The μ's are subject to four equations only, since every b^k is equal to the corresponding a^k. These are

$$\begin{aligned}
&\mu_0 a_0^{k+1} + \mu_1 a_1^{k+1} + \cdots + \mu_k a_k^{k+1} + \mu_{k+1} a_{k+1}^{k+1} + a a_0^k = 0,\\
&\mu_0 b_0^{k+1} + \mu_1 b_1^{k+1} + \cdots + \mu_k b_k^{k+1} + \mu_{k+1} b_{k+1}^{k+1} + a a_0^k = 0,\\
&\mu_0 a_0^k \quad + \mu_1 a_1^k \quad + \cdots + \mu_k a_k^k \qquad\qquad\qquad\qquad = 0,\\
&\qquad \mu_1 a_0^k \quad + \cdots + \mu_k a_{k-1}^k + \mu_{k-1} a_k^k \qquad\qquad = 0.
\end{aligned}$$

The number of arbitrary parameters is therefore $k+3-4$, that is, $k-1$; the number of equations of degree $k+1$ is in general $k-1$, though a linear relation connecting the four equations may increase this number to k. As before, it is already known that it cannot exceed k.

13. The theorem of ascent is the following:

If two equations, satisfied by the coefficients of two curves, have a common derivate, then they have a common source.

So much light is thrown on the important points in the proof by numerical examples, that it seems advisable to preface the formal discussion by two of these, relating to the two principal cases that present themselves.

(i) Let the curves be

$$\begin{aligned}
u &= \cdots + u_7 + x^4y^2 + x^3y^3 + x^2y^4 + x^5 - x^3y^2 + x^2y^3 - y^5 + x^4 + y^4 = 0,\\
v &= \cdots + v_7 + x^6 + x^3y^3 + y^6 + x^5 + y^5 + x^2y^2 = 0;
\end{aligned}$$

these have at the origin quadruple points, with tangents that are different for the two curves. There are two equations of the fifth degree, namely,

$$\begin{aligned}
E_1 &= z_1^5 - z_3^5 - z_0^4 + z_2^4 = 0,\\
E_2 &= z_0^5 - z_4^5 - z_0^4 - z_2^4 = 0.
\end{aligned}$$

Forming the derivates of these, we find

$$\begin{aligned}
D_xE_1 &= z_1^4 = 0, & D_yE_1 &= z_0^4 - z_4^4 = 0,\\
D_xE_2 &= z_0^4 - z_4^4 = 0, & D_yE_2 &= -z_3^4 = 0.
\end{aligned}$$

Hence $D_yE_1 = D_xE_2$. It is to be shown that an equation $E = 0$ can be found, satisfied by u and v, such that $E_1 = D_xE$, $E_2 = D_yE$.

It must be borne in mind that any equation can be modified by means of any of the derivates. Thus if E be

$$a_0 z_0^6 + \cdots + \beta_0 z_0^5 + \cdots + \gamma_0 z_0^4 + \cdots$$

we can add to this $\lambda_1 E_1 + \lambda_2 E_2$, so making two of the β's assume arbitrarily chosen values: and then we can modify the γ's in a similar manner. Since the origin is a quadruple point on both u and v, every z below z^4 is zero.

The values of a are to be chosen so that the two derivates of E may agree with E_1 and E_2; and if β_0 and β_5 are then made zero, as also γ_1, γ_3, γ_4, the whole equation can be written

$$z_1^6 - z_5^6 + \beta_1 z_1^5 + \beta_2 z_2^5 + \beta_3 z_3^5 + \beta_4 z_4^5 + \gamma_0 z_0^4 + \gamma_2 z_2^4 = 0.$$

The x-derivate of this, namely,

$$z_1^5 - z_5^5 + \beta_1 z_1^4 + \beta_2 z_2^4 + \beta_3 z_3^4 + \beta_4 z_4^4,$$

when modified by means of derivates, if necessary, is to be the same as

$$z_1^5 - z_5^5 - z_0^4 + z_2^4.$$

Since $z_1^4 = 0$, $z_3^4 = 0$, and $z_4^4 = z_0^4$, this requires $\beta_2 = 1$, $\beta_4 = -1$.

Similarly

$$z_0^5 - z_4^5 + \beta_1 z_0^4 + \beta_2 z_1^4 + \beta_3 z_2^4 + \beta_4 z_3^4$$

is to be the same as

$$z_0^5 - z_4^5 - z_0^4 - z_2^4,$$

and therefore $\beta_1 = -1$, $\beta_3 = -1$.

Comparison of the derivates gives no information about γ_0 and γ_2; thus the desired source is

$$z_1^6 - z_5^6 - z_1^5 + z_2^5 - z_3^5 - z_4^5 + \gamma_0 z_0^4 + \gamma_2 z_2^4 = 0.$$

Up to this point the result is applicable to any curves for which the given equations hold. The two parameters in the terms of degree 4 make it possible to apply the equation to any two such curves. For the two given curves the necessary values are $\gamma_0 = 2$, $\gamma_2 = 0$, and the desired source is the equation of the 6th degree

$$z_1^6 - z_5^6 - z_1^5 + z_2^5 - z_3^5 - z_4^5 + 2z_0^4 = 0.$$

The set of equations is in this case regular, and the diagram is

$$\begin{array}{ll} E \; . \; . \; . \; . \; . & \text{of degree } 6, \\ E_0^1 \; E_1^1 \; . \; . \; . & \text{of degree } 5, \\ E_0^2 \; E_1^2 \; E_2^2 \; . \; . & \text{of degree } 4, \\ E_0^3 \; E_1^3 \; E_2^3 \; E_3^3 \; . & \text{of degree } 3. \end{array}$$

The presence, in the source determined without reference to the particular curves, of two arbitrary parameters in the terms of lowest degree k, is what characterizes the solution.

(ii) For the curves

$$u = \cdots + x^6 + x^4y^2 - x^3y^3 + y^6 + 10\,x^5 + 5\,x^4y + 20\,x^3y^2 + 5\,x^2y^3 + 10\,xy^4$$
$$+ 5\,y^5 + 5\,x^4 + 8\,x^3y + 3\,x^2y^2 + xy^3 + 2\,x^3 + 7\,x^2y + 7\,xy^2 + 2\,y^3 = 0\,,$$
$$v = \cdots + 6\,x^6 - 5\,x^5y + x^4y^2 - x^3y^3 - 10\,x^2y^4 - 5\,xy^5 - 6\,y^6 - 8\,x^5 - 48\,x^4y$$
$$- 12\,x^3y^2 + x^2y^3 + 8\,xy^4 + 3\,y^5 + x^4 - 4\,x^3y - 4\,x^2y^2 + 4\,xy^3 + 2\,y^4$$
$$+ 2\,x^3 + 7\,x^2y + 7\,xy^2 + 2\,y^3 = 0\,,$$

there are satisfied two equations of the fifth degree with all their derivates. These are

$$E_1 = 32\,z_0^5 - 56\,z_1^5 + 88\,z_2^5 - 144\,z_3^5 + 252\,z_4^5 - 466\,z_5^5$$
$$+ 37\,z_0^4 + 5\,z_1^4 + 7\,z_2^4 + 5\,z_3^4 - 169\,z_4^4 - \tfrac{1521}{2}\,z_0^3 = 0\,,$$
$$E_2 = -56\,z_0^5 + 88\,z_1^5 - 144\,z_2^5 + 252\,z_3^5 - 466\,z_4^5 + 893\,z_5^5$$
$$- 181\,z_0^4 + 6\,z_1^4 + 8\,z_2^4 + 10\,z_3^4 + 142\,z_4^4 + 1379\,z_0^3 = 0\,.$$

The identity of the y-derivate of the first with the x-derivate of the second appears when the terms of the third degree are modified by means of the values for z_0^3, z_1^3, z_2^3, z_3^3 given by the next derivates. These derivates,

$$\begin{aligned}
D_x^2E_1 &= 32\,z_0^3 - 56\,z_1^3 + 88\,z_2^3 - 144\,z_3^3 = 0\,,\\
D_xD_yE_1 = D_x^2E_2 &= -56\,z_0^3 + 88\,z_1^3 - 144\,z_2^3 + 252\,z_3^3 = 0\,,\\
D_y^2E_1 = D_xD_yE_2 &= 88\,z_0^3 - 144\,z_1^3 + 252\,z_2^3 - 466\,z_3^3 = 0\,,\\
D_y^2E_2 &= -144\,z_0^3 + 252\,z_1^3 - 466\,z_2^3 + 893\,z_3^3 = 0\,,
\end{aligned}$$

are equivalent to three only, for the result of multiplying by $2, 7, 7, 2$ and adding vanishes identically; solving, we have $z_0^3 : z_1^3 : z_2^3 : z_3^3 = 2 : 7 : 7 : 2$.

By means of these values, the derivates of E_1, E_2 can be written

$$\begin{aligned}
D_xE_1 &= 32\,z_0^4 - 56\,z_1^4 + 88\,z_2^4 - 144\,z_3^4 + 252\,z_4^4 + 84\,z_0^3 = 0\,,\\
D_yE_1 = D_xE_2 &= -56\,z_0^4 + 88\,z_1^4 - 144\,z_2^4 + 252\,z_3^4 - 466\,z_4^4 - 122\,z_0^3 = 0\,,\\
D_yE_2 &= 88\,z_0^4 - 144\,z_1^4 + 252\,z_2^4 - 466\,z_3^4 + 893\,z_4^4 + 211\,z_0^3 = 0\,.
\end{aligned}$$

The highest terms in a possible source for these two can be written down at once, and the equation of the source can be taken as

$$E = 32 z_0^6 - 56 z_1^6 + 88 z_2^6 - 144 z_3^6 + 252 z_4^6 - 466 z_5^6 + 893 z_6^6$$
$$+ \beta_0 z_0^5 + \cdots + (z_0^4, z_1^4, \cdots) + \kappa z_0^3 = 0.$$

The two derivates of this are to agree with the given equations; hence

$$\beta_0 z_0^4 + \beta_1 z_1^4 + \cdots + \beta_4 z_4^4,$$

$$\beta_1 z_0^4 + \beta_2 z_1^4 + \cdots + \beta_5 z_4^4,$$

must agree with

$$37 z_0^4 + 5 z_1^4 + 7 z_2^4 + 5 z_3^4 - 169 z_4^4$$

and

$$-181 z_0^4 + 6 z_1^4 + 8 z_2^4 + 10 z_3^4 + 142 z_4^4$$

with the help of the derivates of the fourth degree. This yields the equations

$$\begin{aligned}
\beta_0 &= 37 + 32\lambda_1 - 56\lambda_2 + 88\lambda_3, \\
\beta_1 &= 5 - 56\lambda_1 + 88\lambda_2 - 144\lambda_3 = -181 + 32\lambda_1' - 56\lambda_2' + 88\lambda_3', \\
\beta_2 &= 7 + 88\lambda_1 - 144\lambda_2 + 252\lambda_3 = 6 - 56\lambda_1' + 88\lambda_2' - 144\lambda_3', \\
\beta_3 &= 5 - 144\lambda_1 + 252\lambda_2 - 466\lambda_3 = 8 + 88\lambda_1' - 144\lambda_2' + 252\lambda_3', \\
\beta_4 &= -169 + 252\lambda_1 - 466\lambda_2 + 893\lambda_3 = 10 - 144\lambda_1' + 252\lambda_2' - 466\lambda_3', \\
\beta_5 & \qquad = 142 + 252\lambda_1' - 66\lambda_2' + 893\lambda_3'.
\end{aligned}$$

The four equations in the λ's and λ''s, obtained by equating the two values for β_1, β_2, β_3 and β_4, are equivalent to three only, on account of the relation already noted in forming the derivates, which holds also for the numerical part of the equations now in question. Moreover, any two relations in the β's can be imposed arbitrarily, inasmuch as the source can be modified by the addition of linear multiples of E_1 and E_2. For simplicity, let these relations be such that $\lambda_2' = 0$, $\lambda_3' = 0$; the three equations then give

$$\lambda_1 = -\tfrac{8375}{80} + \tfrac{7}{8}A, \quad \lambda_2 = -\tfrac{10987}{80} + \tfrac{7}{8}A, \quad \lambda_3 = -\tfrac{1677}{40} + \tfrac{1}{4}A,$$

where A is written for $-4\lambda_1'$; and the β's are consequently

$$\begin{aligned}
\beta_0 &= 688\tfrac{1}{2} + A, & \beta_2 &= 6 + 14A, & \beta_4 &= 10 + 36A, \\
\beta_1 &= -181 - 8A, & \beta_3 &= 8 - 22A, & \beta_5 &= 142 - 63A.
\end{aligned}$$

The source can now be written

$$
\begin{aligned}
32 z_0^6 - 56 z_1^6 + 88 z_2^6 - 144 z_3^6 + 252 z_4^6 - 466 z_5^6 + 893 z_6^6 & \\
+ (688\tfrac{1}{2} z_0^5 - 181 z_1^5 + 6 z_2^5 + 8 z_3^5 + 10 z_4^5 + 142 z_5^5) & \\
+ A (z_0^5 - 8 z_1^5 + 14 z_2^5 - 22 z_3^5 + 36 z_4^5 - 63 z_5^5) & \\
+ \gamma_0 z_0^4 + \gamma_4 z_4^4 + \kappa z_0^3 = 0 &,
\end{aligned}
$$

in which the terms in z^4 have been modified by the addition of linear multiples of the three independent derivates of the fourth degree, chosen so as to make $\gamma_1 = 0$, $\gamma_2 = 0$, $\gamma_3 = 0$; and comparing the derivates of this with E_1 and E_2, increased by $\lambda_1 D_x E_1 + \lambda_2 D_x E_2 + \lambda_3 D_y E_2$ and $-\frac{1}{4} A D_x E_1$, we find that $\gamma_0 z_0^3$ must agree with $-\frac{1521}{2} z_0^3 + \lambda_1 84 z_0^3 - \lambda_2 122 z_0^3 + \lambda_3 211 z_0^3$, and $\gamma_4 z_3^3$ with $1379 z_0^3 - \frac{1}{4} A\, 84 z_0^3$.

The first gives

$$
\begin{aligned}
\gamma_0 = -\tfrac{1521}{2} + (-\tfrac{8375}{80} + \tfrac{7}{8} A) 84 + (\tfrac{10987}{80} - \tfrac{7}{8} A) 122 & \\
+ (-\tfrac{16777}{40} + \tfrac{1}{4} A) 211 = -\tfrac{6581}{4} + \tfrac{39}{2} A &;
\end{aligned}
$$

and the second, by help of the relation $z_0^3 = z_3^3$, gives $\gamma_4 = 1379 - 21A$.

Thus the desired source is of the form

$$
\begin{aligned}
& 32 z_0^6 - 56 z_1^6 + 88 z_2^6 - 144 z_3^6 + 252 z_4^6 - 466 z_5^6 + 893 z_6^6 \\
& + (688\tfrac{1}{2} z_0^5 - 181 z_1^5 + 6 z_2^5 + 8 z_3^5 + 10 z_4^5 + 142 z_5^5 - \tfrac{6581}{4} z_0^4 + 1379 z_4^4) \\
& + A (z_0^5 - 8 z_1^5 + 14 z_2^5 - 22 z_3^5 + 36 z_4^5 - 63 z_5^5 + \tfrac{39}{2} z_0^4 - 21 z_4^4) + \kappa z_0^3 = 0.
\end{aligned}
$$

As before, this involves two parameters, which must be determined by means of the two given curves. The values found are $A = \frac{1}{2}$, $\kappa = -11$, so that the two given equations are derived from the source

$$
\begin{aligned}
32 z_0^6 - 56 z_1^6 + 88 z_2^6 - 144 z_3^6 + 252 z_4^6 - 466 z_5^6 + 893 z_6^6 + 689 z_0^5 - 185 z_1^5 & \\
+ 13 z_2^5 - 3 z_3^5 + 28 z_4^5 + \tfrac{221}{2} z_5^5 - \tfrac{3271}{2} z_0^4 + \tfrac{2737}{2} z_4^4 - 11 z_0^3 = 0 &
\end{aligned}
$$

The equations in this example are

$$
\begin{array}{llll}
E & . \; . \; . \; . \; . & & \text{of degree } 6, \\
E_0^1 & E_1^1 & . \; . \; . & \text{of degree } 5, \\
E_0^2 & E_1^2 & E_2^2 \; . & \text{of degree } 4, \\
E_0^3 & E_1^3 & E_2^3 \; . & \text{of degree } 3;
\end{array}
$$

the solution is characterized by the presence of two parameters in the source, so far as determined without reference to the particular curves, but there is now

the difference that on y one of these is in the terms of degree k; the first arbitrary parameter appears in connection with the suspension of the aw of unit increase in the number of the equations obtained at the successive stages of the derivation.

14. The general algebraic proof that *if two equations have a common derivate they have a common source for two curves,* depends on obtaining the equation of a source involving two arbitrary parameters.

The two given equations of degree $p-1$ can be reduced to the form

$$E_1 = a_0 z_0^{p-1} + a_1 z_1^{p-1} + a_2 z_2^{p-1} + \cdots + a_{p-1} z_{p-1}^{p-1} + b_0 z_0^{p-2} + b_1 z_1^{p-2} + \cdots + c_0 z_0^{p-3} + \cdots = 0,$$

$$E_2 = a_1 z_0^{p-1} + a_2 z_1^{p-1} + a_3 z_2^{p-1} + \cdots + a_p z_{p-1}^{p-1} + b_1 z_0^{p-2} + b_2 z_1^{p-2} + \cdots + c_1 z_0^{p-3} + \cdots = 0,$$

in which the agreement of $D_y E_1$ with $D_x E_2$ is obvious. For if the second is given as

$$E' = a_1' z_0^{p-1} + a_2' z_1^{p-1} + \cdots + a_p' z_{p-1}^{p-1} + b_1' z_0^{p-2} + b_2' z_1^{p-2} + \cdots + c_1' z_0^{p-3} + \cdots = 0,$$

the fact that the four derivates reduce to three shows that there is a linear relation which may be written

$$\lambda D_x E_1 + \mu D_y E_1 + \lambda' D_x E' + \mu' D_y E' = 0,$$

that is,

$$D_x(\lambda E_1 + \lambda' E') + D_y(\mu E_1 + \mu' E') = 0;$$

thus if we take instead of the given equations these two linear combinations we have the desired form. Even if the given equations are specialized so that they yield only two derivates instead of three, they can be written in this form; the coefficients are now subject to the conditions obtained by expressing that the three equations

$$a_i z_0^{p-2} + \cdots + b_i z_0^{p-3} + \cdots + c_i z_0^{p-1} + \cdots = 0 \qquad (i=0,\,1,\,2)$$

are equivalent to two only, namely, the vanishing of the determinants

$$\left\| \begin{matrix} a_0\, a_1\, a_2 \cdots a_{p-2}\, b_0\, b_1 \cdots b_{p-3}\, c_0 \cdots \\ a_1\, a_2\, a_3 \cdots a_{p-1}\, b_1\, b_2 \cdots b_{p-2}\, c_1 \cdots \\ a_2\, a_3\, a_4 \cdots a_p \quad b_2\, b_3 \cdots b_{p-1}\, c_2 \cdots \end{matrix} \right\|.$$

If $(a_h)^q$ be written for $a_h z_0^q + a_{h+1} z_1^q + a_{h+2} z_2^q + \cdots + a_{h+q} z_q^q$, the given equations become

$$E_1 = (a_0)^{p-1} + (b_0)^{p-2} + (c_0)^{p-3} + \cdots = 0,$$

$$E_2 = (a_1)^{p-1} + (b_1)^{p-2} + (c_1)^{p-3} + \cdots = 0.$$

Let the source to be found be written

$$E = (a_0)^p + (\beta_0)^{p-1} + (\gamma_0)^{p-2} + \cdots + (\tau_0)^{k+1} + (\ )^k = 0.$$

For the determination of the coefficients we have the equations obtained by expressing the identity of the two derivates of E with the given equations, modified by the addition of multiples of lower derivates. These will give for (a), (β), (γ), etc., values involving a certain number of arbitrary parameters. Inasmuch however as the source E may itself be modified in the same manner, its most general expression must involve a certain number of non-significant parameters. The terms $(\beta_0)^{p-1}$, for instance, may be modified by the addition of $B_1 E_1 + B_2 E_2$; to the B's arbitrary values may be assigned; we may, e. g., if we choose, make β_0 and $\beta_{p-1} = 0$. The β's, then, must involve in their expression two non-significant parameters; and if they involve only two in all, both are non-significant, and the β's are looked upon as absolutely determinate.

The agreement of the two derivates of E with the given equations requires only that $D_x E$ and $D_y E$ shall be linear functions of E_1 and E_2, that is, that

$$D_x E = \lambda_1 E_1 + \lambda_2 E_2, \quad D_y E = \lambda_1' E_1 + \lambda_2' E_2.$$

Hence we must have

$$\begin{aligned} a_0 &= \lambda_1 a_0 + \lambda_2 a_1, \\ a_i &= \lambda_1 a_i + \lambda_2 a_{i+1} = \lambda_1' a_{i-1} + \lambda_2' a_i \qquad (i = 1, \cdots, p-1), \\ a_p &= \lambda_1' a_{p-1} + \lambda_2' a_p. \end{aligned}$$

The equations for λ_1, λ_2, λ_1', λ_2' are therefore

$$\lambda_1' a_{i-1} + (\lambda_2' - \lambda_1) a_i - \lambda_2 a_{i+1} = 0 \qquad (i = 1, \cdots, p-1)$$

where p is necessarily $\geqq 4$. Unless the determinants of the third order

$$\left\|\begin{matrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{p-2} \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_{p-1} \\ a_2 & a_3 & a_4 & a_5 & \cdots & a_p \end{matrix}\right\|$$

all vanish, these equations give

$$\lambda_1' = 0, \quad \lambda_2' = \lambda_1, \quad \lambda_2 = 0,$$

from which $(a_0)^p = \lambda_1 (a_0)^p$, where the λ_1 may of course be taken unity. In this case $D_x E = E_1$, $D_y E = E_2$.

If however these determinants do vanish, that is, if

$$a_{i+1} = \eta_1 a_{i-1} + \eta_2 a_i \qquad (i=1, \cdots, p-1),$$

the equations reduce to

$$(\lambda_1' - \eta_1\lambda_2) a_{i-1} \quad + (\lambda_2' - \lambda_1 - \eta_2\lambda_2) a_i = 0 \quad (i=1, \cdots, p-1),$$

from which

$$\lambda_1' = \eta_1\lambda_2, \quad \lambda_2' = \lambda_1 + \eta_2\lambda_2.$$

Hence

$$a_i = \lambda_1 a_i + \lambda_2 a_{i+1}, \qquad (i=0, \cdots, p),$$

where λ_1, λ_2 are arbitrary parameters, and a_{p+1} is written for $\eta_1 a_{p-1} + \eta_2 a_p$.

Thus if the two given equations yield only two derivates the terms of highest degree in the source can be written $A_1(a_0)^p + A_2(a_1)^p$; this special case is considered in § 19. In the general case the coefficients a are determined without ambiguity, and the source is

$$E = (a_0)^p + (\beta_0)^{p-1} + (\gamma_0)^{p-2} + \cdots = 0.$$

15. The next step in the comparison of the two derivates of E with E_1 and E_2 shows that the β's must satisfy

$$\begin{aligned}
\beta_0 &= b_0 + \lambda_1 a_0 + \lambda_2 a_1 + \lambda_3 a_2, \\
\beta_i &= b_i + \lambda_1 a_i + \lambda_2 a_{i+1} + \lambda_3 a_{i+2} = b_i + \lambda_1' a_{i-1} + \lambda_2' a_i + \lambda_3' a_{i+1} \quad (i=1, \cdots, p-2), \\
\beta_{p-1} &= b_{p-1} + \lambda_1' a_{p-2} + \lambda_2' a_{p-1} + \lambda_3' a_p,
\end{aligned}$$

where the λ's and λ''s are to be determined. The equations obtained from the double values for $\beta_1, \cdots, \beta_{p-2}$ give

$$\lambda_1' a_{i-1} + (\lambda_2' - \lambda_1) a_i + (\lambda_3' - \lambda_2) a_{i+1} - \lambda_3 a_{i+2} = 0 \quad (i=1, \cdots, p-2).$$

Unless

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_{p-3} \\ a_1 & a_2 & \cdots & a_{p-2} \\ a_2 & a_3 & \cdots & a_{p-1} \\ a_3 & a_4 & \cdots & a_p \end{vmatrix} = 0,$$

(in which case the four derivates of degree $p-3$ reduce to three only) these equations give

$$\lambda_1' = 0, \quad \lambda_2' = \lambda_1, \quad \lambda_3' = \lambda_2, \quad \lambda_3 = 0,$$

from which

$$\beta_i = b_i + \lambda_1 a_i + \lambda_2 a_{i+1} \qquad (i = 0, \cdots, p-1),$$

where λ_1, λ_2 are arbitrary. The β's, however, involve two non-significant parameters, since, as already pointed out, the source may be modified by the addition of $B_1 E_1$ and $B_2 E_2$; we may therefore choose λ_1, λ_2 arbitrarily, for instance, $\lambda_1 = 0$, $\lambda_2 = 0$, values which have the advantage of simplifying the work, inasmuch as no multiples of the three derivates of E_1 and E_2 are now used. We have now $\beta_0 = b_0$, etc., and the source is

$$(a_0)^p + (b_0)^{p-1} + (\gamma_0)^{p-2} + (\delta_0)^{p-3} + \cdots = 0.$$

The derivates of this, namely,

$$(a_i)^{p-1} + (b_i)^{p-2} + (\gamma_i)^{p-3} + (\delta_i)^{p-4} + \cdots = 0 \qquad (i = 0, 1),$$

are to agree with

$$(a_i)^{p-1} + (b_i)^{p-2} + (c_i)^{p-3} + (d_i)^{p-4} + \cdots = 0 \qquad (i = 0, 1),$$

with the help of the four derivates of degree $p-3$,

$$(a_i)^{p-3} + (b_i)^{p-4} + \cdots = 0 \qquad (i = 0, 1, 2, 3);$$

hence

$$\gamma_0 = c_0 + \lambda_1 a_0 + \lambda_2 a_1 + \lambda_3 a_2 + \lambda_4 a_3,$$

$$\gamma_i = c_i + \lambda_1 a_i + \lambda_2 a_{i+1} + \lambda_3 a_{i+2} + \lambda_4 a_{i+3} = c_i + \lambda_1' a_{i-1} + \lambda_2' a_i + \lambda_3' a_{i+1} + \lambda_4' a_{i+2} \qquad (i = 1, \cdots, p-3),$$

$$\gamma_{p-2} = c_{p-2} + \lambda_1' a_{p-3} + \lambda_2' a_{p-2} + \lambda_3' a_{p-1} + \lambda_4' a_p.$$

Then the equations for the λ's and λ''s are

$$\lambda_1' a_{i-1} + (\lambda_2' - \lambda_1) a_i + (\lambda_3' - \lambda_2) a_{i+1} + (\lambda_4' - \lambda_3) a_{i+2} - \lambda_4 a_{i+3} = 0 \qquad (i = 1, \cdots, p-3);$$

hence, unless

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_{p-4} \\ a_1 & a_2 & \cdots & a_{p-3} \\ a_2 & a_3 & \cdots & a_{p-2} \\ a_3 & a_4 & \cdots & a_{p-1} \\ a_4 & a_5 & \cdots & a_p \end{vmatrix} = 0,$$

we find

$$\lambda_1' = 0, \quad \lambda_2' = \lambda_1, \quad \lambda_3' = \lambda_2, \quad \lambda_4' = \lambda_3, \quad \lambda_4 = 0.$$

The expressions for the γ's involve therefore the three parameters $\lambda_1, \lambda_2, \lambda$, since however there must be three non-significant parameters, we may choose $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$. Then $\gamma_0 = c_0, \gamma_1 = c_1$, etc., and the source is

$$E = (a_0)^p + (b_0)^{p-1} + (c_0)^{p-2} + \text{etc.} \cdots = 0.$$

This argument is perfectly general; the coefficients in the terms of degree $p - m$ are linearly expressed in terms of the quantities

$$\lambda_1, \lambda_2, \cdots, \lambda_{m+2}; \quad \lambda'_1, \lambda'_2, \cdots, \lambda'_{m+2},$$

given by the equations

$$\lambda'_1 a_{i-1} + (\lambda'_2 - \lambda_1) a_i + (\lambda'_3 - \lambda_2) a_{i+1} + \cdots$$
$$+ (\lambda'_{m+2} - \lambda_{m+1}) a_{i+m} - \lambda_{m+2} a_{i-m-1} = 0 \qquad (i = 1, \cdots, p - m - 1);$$

here there are $m + 3$ quantities to be determined by means of $p - m - 1$ equations. Leaving the terms of degree k or $k + 1$ to be considered separately, we have $p - m > k$, and $m < k - 2$, consequently

$$p - m - 1 > k - 1, \quad m + 2 < k,$$

that is, $p - m - 1 \geqq m + 3$. Every combination of λ's and λ''s is therefore zero. Hence, unless

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_{p-m-2} \\ a_1 & a_2 & \cdots & \vdots \\ \vdots & & & \vdots \\ a_{m+2} & a_{m+3} & \cdots & a_p \end{vmatrix} = 0,$$

(that is, unless the $m + 3$ derivates of degree $p - m - 2$ reduce to $m + 2$ of this degree), the coefficients of the terms of degree $p - m$ involve linearly parameters whose number, $m + 1$, is precisely the number of non-significant parameters that must enter into these coefficients: these may all be taken zero, with the result that the coefficients of the terms of degree $p - m$ in the source are those immediately given by the terms of degree $p - m - 1$ in the two given equations.

If now there is no reduction in the number of derivates at any stage, the scheme of equations is regular, and there are therefore $k - 1$ equations of degree k, by means of which the values of $z_0^k, z_1^k, \cdots, z_k^k$ are all expressed in terms of any two, e. g., of z_0^k and z_k^k. The terms of degree k in the source can then be written $A_1 z_0^k + A_2 z_k^k$, and as no information is given about A_1, A_2 by comparison with the derivates, since these terms disappear on derivation, these are *two arbitrary parameters entering into the equation of the source*,

$$E = (a_0)^p + (b_0)^{p-1} + (c_0)^{p-2} + \cdots + (k_0)^{k+1} + A_1 z_0^k + A_2 z_k^k = 0.$$

16. If there are k equations of degree k, so that $z_0^k, z_1^k, \cdots, z_k^k$ are all expressed in terms of any one, the scheme is not strictly regular, even though it be regular

up to this point. In this case $p = 2k - 1$. The terms of degree k in the source are now reducible to Az_0^k, involving only one parameter. But the equations for $\tau_0, \tau_1, \cdots, \tau_{k+1}$ are

$$\begin{aligned}
\tau_0 &= t_0 + \lambda_1 a_0 + \cdots + \lambda_k a_{k-1} \\
\tau_i &= t_i + \lambda_1 a_i + \cdots + \lambda_k a_{i+k-1} = t_i + \lambda_1' a_{i-1} + \cdots + \lambda_k' a_{i+k-2} \\
&\qquad\qquad (i = 1, \cdots, k), \\
\tau_{k+1} &= t_{k+1} + \lambda_1' a_k + \cdots + \lambda_k' a_{2k-1}.
\end{aligned}$$

Hence we have for the λ's and λ''s the k equations:

$$\lambda_1' a_{i-1} + (\lambda_2' - \lambda_1) a_i + \cdots + (\lambda_k' - \lambda_{k-1}) a_{i+k-2} - \lambda_k a_{i+k-1} = 0 \quad (i = 1, \cdots, k),$$

involving the λ's, etc., in $k + 1$ combinations. By means of these equations the quantities λ' are expressed in terms of the k parameters $\lambda_1, \cdots, \lambda_k$. The terms $(\tau_0)^{k+1}$ in the source involve $k - 1$ non-significant parameters, since there are $k - 1$ derivates of degree $k + 1$, and accordingly there is one significant parameter involved in these terms. As before, *there are two arbitrary parameters in the equation of the source*, which is now

$$E = (a_0)^p + (b_0)^{p-1} + \cdots + (t_0)^{k+1} + A_1 E + A_2 z_0^k = 0,$$

where E is of degree $k + 1$.

17. *Partial agreement of the derivates.*—At every stage in the determination of the coefficients in the source the possibility of an exception presents itself. This exception depends on the existence of a relation among the a's, by means of which the $m + 3$ derivates of degree $p - m - 2$ are reduced to $m + 2$. Let the terms in the source be $(\rho_0)^{p-m}$; the agreement of $(\rho_0)^{p-m-1}$ and $(\rho_1)^{p-m-1}$ with $(r_0)^{p-m-1}$ and $(r_1)^{p-m-1}$ gives the equations

$$\begin{aligned}
\rho_0 &= r_0 + \lambda_1 a_0 + \cdots + \lambda_{m+2} a_{m+1}, \\
\rho_i &= r_i + \lambda_1 a_i + \cdots + \lambda_{m+2} a_{i+m+1} = r_i + \lambda_1' a_{i-1} + \cdots + \lambda_{m+2}' a_{i+m} \\
&\qquad\qquad (i = 1, \cdots, p - m - 1), \\
\rho_{p-m} &= r_{p-m} + \lambda_1' a_{p-m-1} + \cdots + \lambda_{m+2}' a_p;
\end{aligned}$$

the λ's and λ''s are therefore determined by $p - m - 1$ equations of the usual type,

$$\lambda_1' a_{i-1} + (\lambda_2' - \lambda_1) a_i + \cdots + (\lambda_{m+2}' - \lambda_{m+1}) a_{i+m} - \lambda_{m+2} a_{i+m+1} = 0 \quad (i = 1, \cdots, p - m - 1),$$

in which the quantities appear in $m + 3$ combinations. But by hypothesis

$$a_{i+m+1} = \eta_1 a_{i-1} + \eta_2 a_i + \eta_3 a_{i+1} + \cdots + \eta_{m+2} a_{i+m} \quad (i = 1, \cdots, p - m - 1);$$

hence the equations for the λ's and λ''s assume the form

$$(\lambda'_1 - \eta_1 \lambda_{m+2}) a_{i-1} + (\lambda'_2 - \lambda_1 - \eta_2 \lambda_{m+2}) a_i + \cdots + (\lambda'_{m+2} - \lambda_{m+1} - \eta_{m+2} \lambda_{m+2}) a_{i+m} = 0$$
$$(i = 1, \cdots, p - m - 1).$$

The $m + 2$ combinations of the λ's and λ''s here involved, equated to zero, leave the λ's arbitrary, the λ''s being expressed in terms of these. The ρ's consequently involve $m + 2$ arbitrary parameters. The number of non-significant parameters being $m + 1$, it is seen that one significant parameter appears in the terms of degree $p - m$ in the source, in consequence of the reduction from $m + 3$ to $m + 2$ of the number of derivates of degree $p - m - 2$. If zero values be chosen for $m + 1$ parameters, e. g., for $\lambda_1, \lambda_2, \cdots, \lambda_{1+m}$, the expressions for the ρ's become

$$\rho_i = r_i + \lambda a_{i+m+1} \qquad (i = 0, \cdots, p - m),$$

where a_{p+1} is written for $\eta_1 a_{p-m-1} + \eta_2 a_{p-m} + \cdots$, and λ for λ_{m+2}.

Now the relations connecting the a's, namely,

$$\left\| \begin{matrix} a_0 & a_1 & \cdots a_{p-m-2} \\ a_1 & a_2 & \cdots a_{p-m-1} \\ a_2 & & \\ \vdots & & \\ a_{m+2} & a_{m+3} & \cdots a_p \end{matrix} \right\| = 0,$$

prove only that the highest terms can be eliminated from the derivates of degree $p - m - 2$. If these same relations hold for the b's, c's, etc., then denoting the derivates by $D_0, D_1, \cdots, D_{m+2}$, we have

$$D_{m+2} = \eta_1 D_0 + \eta_2 D_1 + \cdots + \eta_{m+2} D_{m+1},$$

and the set of equations has become stationary. If however these relations do not hold for every set of coefficients (b), (c), $\cdots$, then for D_{m+2} is substituted the equation of lower degree

$$D_{m+2} - (\eta_1 D_0 + \eta_2 D_1 + \cdots \eta_{m+2} D_{m+1}) = 0.$$

If the b's are not subject to the relations, the degree of this equation is $p - m - 3$; if the b's are subject, but the c's not, the degree is $p - m - 4$, and so on. Let this depressed equation be denoted by

$$(A_{m+2})^{p-m-2} + (B_{m+2})^{p-m-3} + \cdots = 0,$$

and let this be inserted at the end of the derivates of degree $p - m - 2$. The equations so far as now required are represented in the accompanying table, in which the degree of the terms in any column is given at the head of the column.

	$p-m$	$p-m-1$	$p-m-2$	$p-m-3$	$p-m-4$	
Source.	$\cdots+(\rho_0)$	$+(\sigma_0)$	$+(\tau_0)$	$+(\xi_0)$	$+(\theta_0)$	$+\cdots$
Given equations.	$\cdots+(q_0)$	$+(r_0)$	$+(s_0)$	$+(t_0)$	$+(x_0)$	$+\cdots$
	$\cdots+(q_1)$	$+(r_1)$	$+(s_1)$	$+(t_1)$	$+(x_1)$	$+\cdots$
Derivates of degree $p-m-1$, in number $m+2$.		(a_0)	$+(b_0)$	$+(c_0)$	$+(d_0)$	$+\cdots$
		(a_1)	$+(b_1)$	$+(c_1)$	$+(d_1)$	$+\cdots$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdot$
		(a_{m+1})	$+(b_{m+1})$	$+(c_{m+1})$	$+(d_{m+1})$	$+\cdots$
Derivates of degree $p-m-2$, reduced to $m+2$ in number.			(a_0)	$+(b_0)$	$+(c_0)$	$+\cdots$
			$\vdots$	$\vdots$	$\vdots$	$\cdot$
			(a_{m+1})	$+(b_{m+1})$	$+(c_{m+1})$	$+\cdots$
Derivates of degree $p-m-3$.				(a_0)	$+(b_0)$	$+\cdots$
				$\vdots$	$\cdot$	$\cdot$
				(a_{m+1})	$+(b_{m+1})$	$+\cdots$
				(A_{m+2})	$+(B_{m+2})$	$+\cdots$
Derivates of degree $p-m-4$.					(a_0)	$+\cdots$
					$\vdots$	$\cdot$
					(a_{m+1})	$+\cdots$
					(A_{m+2})	$+\cdots$
					(A_{m+3})	$+\cdots$

For the determination of the coefficients σ, which must involve in their expression $m+2$ non-significant parameters, we have the equations

$$\sigma_0 = s_0 + \sum \lambda b_0 + \kappa_1 a_0 + \cdots + \kappa_{m+2} a_{m+1},$$

$$\begin{aligned}\sigma_i &= s_i + \sum \lambda b_i + \kappa_1 a_i + \cdots + \kappa_{m+2} a_{i+m+1} \\ &= s_i + \sum \lambda' b_{i-1} + \kappa'_1 a_{i-1} + \cdots + \kappa'_{m+2} a_{i+m} \qquad (i = 1, \cdots, p-m-2), \\ \sigma_{p-m-1} &= s_{p-m-1} + \sum \lambda' b_{p-m-2} + \kappa'_1 a_{p-m-2} + \cdots + \kappa'_{m+2} a_{p-1}.\end{aligned}$$

Now

$$\begin{aligned}\sum \lambda b_1 - \sum \lambda' b_0 &= \lambda_{m+2} b_{m+2} - (\lambda'_1 b_0 + \lambda'_2 b_1 + \cdots + \lambda'_{m+2} b_{m+1}) \\ &= \lambda \{ b_{m+2} - (\eta_1 b_0 + \eta_2 b_1 + \cdots + \eta_{m+2} b_{m+1}) \} \\ &= \lambda A_{m+2}.\end{aligned}$$

Hence the equations for κ, κ' are

$$\kappa_1' a_{i-1} + (\kappa_2' - \kappa_1) a_i + \cdots + (\kappa_{m+2}' - \kappa_{m+1}) a_{i+m} - \kappa_{m+2} a_{i+m+1} = \lambda A_{i+m+1}$$
$$(i = 1, \cdots, p - m - 2),$$

that is,

$$(\kappa_1' - \eta_1 \kappa_{m+2}) a_{i-1} + (\kappa_2' - \kappa_1 - \eta_2 \kappa_{m+2}) a_i$$
$$+ \cdots + (\kappa_{m+2}' - \kappa_{m+1} - \eta_{m+2} \kappa_{m+2}) a_{i+m} = \lambda A_{i+m+1}$$
$$(i = 1, \cdots, p - m - 2).$$

Here every combination of κ's and κ''s must be zero, as also the expressions on the right, on account of the number of the equations. Consequently unless every A is zero, in which case the equation of depressed degree does not appear at this stage, we must have $\lambda = 0$. Thus the significant parameter in the terms $(\rho_0)^{p-m}$ turns out to be deceptive: its value is now assigned, so that the ρ's are determinate, and the work proceeds as in the general case. The $m + 2$ arbitrary parameters, $\kappa_1, \cdots, \kappa_{m+2}$, are all non-significant, and can conveniently be taken to be zero, and the σ's are consequently determinate.

If however every A is zero, the equation substituted for D_{m+3} takes its proper place among the derivates of degree $p - m - 4$. (In the table just given the line $(A_{m+2})^{p-m-3} + (B_{m+2})^{p-m-4} + \cdots$ in the derivates of degree $p - m - 3$ and the line $(A_{m+3})^{p-m-4} + \cdots$ in those of degree $p - m - 4$ are to be struck out.) The equations for τ are

$$\tau_0 = t_0 + \sum \lambda c_0 + \nu_1 a_0 + \cdots + \nu_{m+2} a_{m+1},$$

$$\tau_i = t_i + \sum \lambda c_i + \nu_1 a_i + \cdots + \nu_{m+2} a_{i+m+1}$$
$$= t_i + \sum \lambda' c_{i-1} + \nu_1' a_{i-1} + \cdots + \nu_{m+2}' a_{i+m}$$
$$(i = 1, \cdots, p - m - 3),$$

$$\tau_{p-m-2} = t_{p-m-2} + \sum \lambda' c_{p-m-3} + \nu_1' a_{p-m-3} + \cdots + \nu_{m+2}' a_{p-2}.$$

Here

$$\sum \lambda c_1 - \sum \lambda' c_0 = \lambda_{m+2} c_{m+2} - (\lambda_1' c_0 + \lambda_2' c_1 + \cdots + \lambda_{m+2}' c_{m+1})$$
$$= \lambda \{ c_{m+2} - (\eta_1 c_0 + \eta_2 c_1 + \cdots + \eta_{m+2} c_{m+1}) \}$$
$$= \lambda A_{m+2}.$$

Consequently the ν's are determined by the set of equations

$$\nu_1' a_{i-1} + (\nu_2' - \nu_1) a_i + \cdots + (\nu_{m+2}' - \nu_{m+1}) a_{i+m} - \nu_{m+2} a_{i+m+1} = \lambda A_{i+m+1}$$
$$(i = 1, \cdots, p - m - 3),$$

that is, by the set of equations

$$(\nu_1' - \eta_1 \nu_{m+2}) a_{i-1} + (\nu_2' - \nu_1 - \eta_2 \nu_{m+2}) a_i + \cdots = \lambda A_{i+m+1}$$
$$(i = 1, \cdots, p - m - 3).$$

Precisely as before, unless every $A = 0$, it follows that $\lambda = 0$; the parameter λ is only temporary. The τ's contain only non-significant parameters in any case, and the work proceeds exactly as before, the first derivates whose multipliers are not zero being those of the degree where the dropped one reappears. The partial agreement of derivates may occur again and again, but this causes no complication; each temporary parameter which presents itself when the law of unit increase is interrupted has its value determined when this law again comes into operation. Thus we have the result that *partial agreement of the derivates of any degree produces no effect on the parameters in the source,* while absolute agreement, the equations having become a stationary set, is expressed by the occurrence of a single parameter in the terms of degree $p - m$, where the natural derivates of degree $p - m - 2$ are subject to a relation by which their number is diminished by unity.

18. *Stationary set.*—The equations for this case are the same as those of the table given above, with the omission of the lines

$$(A_{m+2})^{p-m-3} + \cdots, \quad (A_{m+2})^{p-m-4} + \cdots, \quad (A_{m+3})^{p-m-4} + \cdots,$$

etc. The equations for σ, already given, lead to the equations for κ, κ', these giving every κ' in terms of κ's, so that there are $m + 2$ parameters, $\kappa_1, \cdots, \kappa_{m+2}$, all non-significant. Hence the σ's are determinate. In like manner the τ's, and all succeeding sets of coefficients, are determinate; the source involves precisely the one parameter λ before we arrive at the terms of degree k. Since we are dealing with equations which have become stationary, there are k equations of degree k, by means of which the terms of degree k in the source can be reduced to $A_2 z_0^k$. The source is therefore

$$E^p + A_1 E^q + A_2 z_0^k = 0,$$

an equation which involves two parameters, as before.

19. One case remains for investigation, that in which *the two given equations yield only two derivates of the next lower degree.* This requires that

$$a_{i+1} = \eta_1 a_{i-1} + \eta_2 a_i \qquad (i = 1, \cdots, p-1).$$

It has been shown (§ 14) that if a_{p+1} be written for $\eta_1 a_{p-1} + \eta_2 a_p$, the terms $(a_0)^p$ in the source are $\lambda_1 (a_0)^p + \lambda_2 (a_1)^p$. Hence

$$D_x E = \lambda_1 E_1 + \lambda_2 E_2, \quad D_y E = \lambda_1 E_2 + \lambda_2 E_3,$$

where

$$E_3 = (a_2)^p + \cdots = \eta_1 E_1 + \eta_2 E_2.$$

Consequently

$$D_y E = \lambda_2 \eta_1 E_1 + (\lambda_1 + \lambda_2 \eta_2) E_2 = \lambda_1' E_1 + \lambda_2' E_2.$$

The equations for β are therefore

$$\beta_0 = \lambda_1 b_0 + \lambda_2 b_1 + \kappa_1 a_0 + \kappa_2 a_1,$$

$$\beta_i = \lambda_1 b_i + \lambda_2 b_{i+1} + \kappa_1 a_i + \kappa_2 a_{i+1} = \lambda_1' b_{i-1} + \lambda_2' b_i + \kappa_1' a_{i-1} + \kappa_2' a_i \qquad (i=1, \cdots, p-2),$$

$$\beta_{p-1} = \lambda_1' b_{p-2} + \lambda_2' b_{p-1} + \kappa_1' a_{p-2} + \kappa_2' a_{p-1}.$$

Hence

$$\kappa_1' a_{i-1} + (\kappa_2' - \kappa_1) a_i - \kappa_2 a_{i+1} = \lambda_1 b_i + \lambda_2 b_{i+1} - \lambda_2 \eta_1 b_{i-1} - (\lambda_1 + \lambda_2 \eta_2) b_i$$
$$= \lambda_2 (b_{i+1} - \eta_1 b_{i-1} - \eta_2 b_i) \qquad (i=1, \cdots, p-2).$$

Since $a_{i+1} = \eta_1 a_{i-1} + \eta_2 a_i$, the equations become, on writing

$$A_{i+1} = b_{i+1} - \eta_1 b_{i-1} - \eta_2 b_i,$$
$$(\kappa_1' - \eta_1 \kappa_2) a_{i-1} + (\kappa_2' - \kappa_1 - \eta_2 \kappa_2) a_i = \lambda_2 A_{i+1} \qquad (i=1, \cdots, p-2).$$

Consequently unless every $A = 0$, that is, unless the b's are subject to the same relation as the a's, we have $\lambda_2 = 0$. That is to say, if the three derivates of the given equations are equivalent to two of degree $p-2$ and one of the next lower degree, the terms of degree p in the source reduce to $(a_0)^p$. If however the b's are subject to this relation, we have $A = 0$. In either case, κ_1' and κ_2' are expressed in terms of κ_1 and κ_2; the β's involve two parameters, both non-significant, and therefore zero if we choose. Precisely the same argument applies to the following sets of terms; and we obtain the result that if

$$D_y E_2 - \eta_1 D_x E_1 - \eta_2 D_y E_1$$

is of degree lower than $p-2$, the parameter in the terms $(a_0)^p$ has its value determined ultimately, so that the source is not affected by this partial agreement; while if this combination of the derivates vanishes, so that the two given equations yield precisely two derivates (as a necessary consequence of which there are only two derivates of any degree), the source is

$$\lambda_1 \{(a_0)^p + \cdots\} + \lambda_2 \{(a_1)^p + \cdots\} + \text{ terms in } (z_0^2, z_1^2, z_2^2) = 0.$$

Since there are two equations of degree two, the ratios $z_0^2 : z_1^2 : z_2^2$ are known; consequently the source is

$$\lambda_1 \{(a_0)^p + \cdots\} + \lambda_2 \{(a_1)^p + \cdots\} + \lambda_3 z_0^2 = 0,$$

and the two parameters are present as in the other cases.

20. The results obtained are that *the general source of degree p involves in its equation two arbitrary parameters.* The source can be written as

$$A_0 E^p + A_1 E^q + A_2 E^k = 0,$$

where

$$\text{(i)} \quad q = k, \qquad \text{(ii)} \quad p > q > k, \qquad \text{(iii)} \quad q = p,\ k = 2.$$

If now the source is to relate to two given curves, u, v, for which the two given equations and all their derivates are satisfied, this supplies two equations for the determination of $A_0 : A_1 : A_2$. It is conceivable that for special curves there may not be a unique determination of these parameters; but as regards cases (i) and (ii) this cannot happen unless the curves are chosen so as to satisfy an equation of degree $\leqq p-1$, not included among those given. For in case (i), if there is still an undetermined parameter, the source

$$A_0 E^p + A_1 E_1^k + A_2 E_2^k = 0$$

becomes

$$A_0 E^p + BE^k = 0,$$

showing that there is an additional equation of degree k, $E^k = 0$, not obtainable from the derivates of the two given equations. Similarly in case (ii) there is an additional equation of degree $\leqq p-1$, which is not included among those given, though its derivates are included. For the arbitrary parameter which presents itself in the determination of the coefficients ρ is the multiplier of an expression

$$a_{m+1} z_0^{p-m} + a_{m+2} z_1^{p-m} + \cdots + a_{p+1} z_{p-m}^{p-m} + \cdots,$$

whose two derivates are $(a_{m+1})^{p-m-1} + \cdots$, and $(a_{m+2})^{p-m-1} + \cdots$. Of these, the first is the last of the $m+2$ derivates of degree $p-m-1$, while in virtue of the relations connecting the a's, the second is linearly expressible in terms of the last two of these $m+2$ derivates. Hence all the derivates of $AE^p + BE^q = 0$ are members of the given system. This new equation is of degree $q = p - m$, while the reduction in the number of derivates occurs at degree $q-2$. As the reduced number is k, there are k derivates of degree $q-1$; beginning with these, the set of k equations is stationary. Now a stationary set may be derived from $k-1$ equations; but it may be derived from k equations, in which case the set has become stationary one stage earlier. Thus such a set of equations as that indicated in the diagram by the * * * may have to be completed, for particular curves, by the equations ○ ○ ○, and again by □ □ □; but these do not arise from the two given equations.

	□			
	○	□		
	*	○	□	
Two given equations →	*	*	○	□
degree q	*	*	*	○
" $q-1$	*	*	*	*
" $q-2$	*	*	*	*.

In both these cases there is strictly only one source of degree p, arising from the given equations. But case (iii) is different. The source is now

$$A_0 E_1^p + A_1 E_2^p + A_2 E^2 = 0 .$$

If the two given curves fail to determine the parameters uniquely, there are two distinct sources of degree p. The explanation is simple enough. The set is stationary, including the two given equations. It may arise from one equation; but it may also arise from two equations, just as a stationary set of k equations may arise from $k-1$ equations or from k equations.

21. In a similar manner can be proved the *generalized theorem of ascent:*

If m equations of degree p yield not more than $m+1$ derivates of degree $p-1$, they arise from not less than $m-1$ equations of degree $p+1$.

Comparing the two derivates of an assumed source

$$(a_0)^{p+1} + (\beta_0)^p + (\gamma_0)^{p-1} + \cdots = 0$$

with general linear functions of the given equations, and modifying the resulting equations by means of relations arising from the fact that the $2m$ derivates of the given equations reduce to $m+1$, we find that the a's involve $m-1$ arbitrary parameters. Assigning any values to these we find that the source is determinate when the values of two parameters in later terms are obtained by means of two given curves. The theorem follows also from the application about to be made of the simple theorem of ascent.

22. *Proof of Theorem* (1).—By means of the theorem of ascent, it can be shown that *any two curves determine a one-set system of equations.* For this we start with the equations of degree k and $k+1$, whose numbers are known to be

(i) $k-1$ and $k-2$, (ii) k and $k-1$, (iii) k and k.

The equations of degree $k+1$ have those of degree k for derivates; hence one of two cases must occur. Either (1) they can be written so that any two successive ones have a common derivate, or (2) they fall into distinct sets, each set giving rise to a part of the derivates of degree k. This last case will be considered separately, and then it will be shown that it does not differ essentially from the first case.

In the first case, by hypothesis, the equations can be written so as to have common derivates; we can therefore ascend step by step, obtaining in general one equation fewer of the next rank, and so on. In (iii) the whole set is sta-

tionary to begin with; the theorem of ascent, applied to the equations in pairs, shows at once that the number of equations of degree $k+2$ will be either $k-1$ or k.

In every event, having arrived at h equations of degree q, we can ascend to their sources of degree $q+1$, obtaining either h or $h-1$ of these. Hence one of two results is inevitable; we arrive finally at a single equation, or else we arrive at a minimum number, h, from which we can ascend indefinitely, obtaining h equations of every higher degree. This however is impossible, for it is shown in § 28 that the number of points of intersection of two curves, falling at the origin, is at least as great as* the number of the equations due to their behavior at the origin.

No equations can exist except those here determined; for since their derivates must be satisfied, those of degree $k+1$ and k are included among the equations used as a starting point for the process of ascent; and the investigations comprised in the proof of the theorem of ascent show that we have obtained every possible source.

23. It has now to be shown that case (2), when the equations break up into sets, does not differ materially from case (1). Suppose, for the sake of generality, that this occurs at degree q; that is, it is to be supposed that m_1 equations of degree q give m_1' of degree $q-1$ as derivates, while the remaining m_2 of degree q give the remaining m_2' of degree $q-1$. If this can happen, the process of ascent applied to each set separately proves the existence of a prime equation for each set. Now if the derivates of degree k, of these two sets, were not entirely independent, the ordinary process of ascent would apply, leading ultimately to one source. Hence the only case to be considered is that where the separation makes itself felt at the very outset, in the equations of degrees k and $k+1$. It has been shown that the number of equations of degree $k+1$ cannot fall short of the number of degree k by more than unity; hence one at least of the two sets into which the equations break up must be stationary. All that has to be proved, in order to establish the applicability of the process of the last section, is that the equations belonging to the two separate sets do not form the complete system, that additional equations result from the combination of the two sets.

Of the equations of degree k, in number $k-1$, let m result from the stationary set, q from the progressive set. Let the number of equations in the stationary set, obtained from this alone, be m at degree p, $m-1$ at degree

*As shown in § 29, these numbers are as a matter of fact equal; but the proof of their equality makes use of the result of the present section, while the proof of the property here stated does not.

$p+1$. There may be equations of the remaining set at this degree (fig. 1), or there may not (fig. 2).

* * * o		* * *
* * * * o o	degree $p+1$	* * * *
* * * * * o o o	degree p	* * * * *
* * * * * o o o o		* * * * *
* * * * * o o o o o		* * * * * o
* * * * * o o o o o o		* * * * * o o
* * * * * o o o o o o o		* * * * * o o o
FIG. 1.		FIG. 2.

It is sufficient to prove that there exists another equation of degree $p+1$, not resulting from either set alone.

If the source of the stationary set be written

$$(a_0)^P + (b_0)^{P-1} + (c_0)^{P-2} + \cdots = 0,$$

the $m-1$ equations of degree $p+1$ are obtained in the form

$$(a_i)^{p+1} + (b_i)^p + (c_i)^{p-1} + \cdots + (h_i)^{k+2} + (j_i)^{k+1} + (k_i)^k = 0 \quad (i = 0, \cdots, m-2).$$

The reduction of the $m+1$ derivates of the degree $p-1$ to m depends on the vanishing of the determinants

$$\left\Vert \begin{matrix} a_0 & a_1 & \cdots a_{p-1} & b_0 \cdots b_{p-2} & c_0 \cdots h_0 \cdots h_k \\ a_1 & a_2 & \cdots a_p & b_1 \cdots b_{p-1} & c_1 \cdots h_1 \cdots h_{k+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_m & a_{m+1} & \cdots a_{p+m-1} & b_m \cdots b_{p+m-2} & c_m \cdots h_m \cdots h_{k+m} \end{matrix} \right\Vert,$$

or, in the form here required, on the existence of linear relations

$$\begin{aligned} a_m &= \eta_1 a_0 + \eta_2 a_1 + \eta_3 a_2 + \cdots + \eta_m a_{m-1}, \\ a_{m+1} &= \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3 + \cdots + \eta_m a_m, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ h_{m+k} &= \eta_1 h_k + \eta_2 h_{k+1} + \eta_3 h_{k+2} + \cdots + \eta_m h_{k+m}. \end{aligned}$$

It is to be noted that a's, b's, $\cdots$, h's can be built up with any suffix by this law of formation.

The proof of the existence of another equation of degree $p+1$ is most easily followed by reference to the accompanying table, in which the accented letters refer to the progressive set, whose source is

$$(a'_0)^{P'} + (b'_0)^{P'-1} + \cdots = 0.$$

$p+1$	p	$p-1$	$p-2$		$k+2$	$k+1$	k
(a_0)	$+(b_0)$	$+(c_0)$	$+(d_0)$	$+\ \ldots$	$+(h_0)$	$+(j_0)$	$+(k_0)$
(a_1)	$+(b_1)$	$+(c_1)$	$+(d_1)$	$+\ \ldots$	$+(h_1)$	$+(j_1)$	$+(k_1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
(a_{m-2})	$+(b_{m-2})$	$+(c_{m-2})$	$+(d_{m-2})$	$+\ \ldots$	$+(h_{m-2})$	$+(j_{m-2})$	$+(k_{m-2})$
(a'_0)	$+(b'_0)$	$+(c'_0)$	$+(d'_0)$	$+\ \ldots$	$+(h'_0)$	$+(j'_0)$	$+(k'_0)$
(a'_1)	$+(b'_1)$	$+(c'_1)$	$+(d'_1)$	$+\ \ldots$	$+(h'_1)$	$+(j'_1)$	$+(k'_1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
(a'_{n-2})	$+(b'_{n-2})$	$+(c'_{n-2})$	$+(d'_{n-2})$	$+\ \ldots$	$+(h'_{n-2})$	$+(j'_{n-2})$	$+(k'_{n-2})$
	(a_0)	$+(b_0)$	$+(c_0)$	$+\ \ldots$	$+(g_0)$	$+(h_0)$	$+(j_0)$
	$\vdots$	$\vdots$	$\vdots$				
	(a_{m-1})	$+(b_{m-1})$	$+(c_{m-1})$	$+\ \ldots$	$+(g_{m-1})$	$+(h_{m-1})$	$+(j_{m-1})$
	(a'_0)	$+(b'_0)$	$+(c'_0)$	$+\ \ldots$	$+(g'_0)$	$+(h'_0)$	$+(j'_0)$
	$\vdots$	$\vdots$	$\vdots$				
	(a'_{n-1})	$+(b'_{n-1})$	$+(c'_{n-1})$	$+\ \ldots$	$+(g'_{n-1})$	$+(h'_{n-1})$	$+(j'_{n-1})$
		(a_0)	$+(b_0)$	$+\ \ldots$		$+(g_0)$	$+(h_0)$
		$\vdots$				$\vdots$	
		(a_{m-1})	$+(b_{m-1})$	$+\ \ldots$		$+(g_{m-1})$	$+(h_{m-1})$
		(a'_0)	$+(b'_0)$	$+\ \ldots$		$+(g'_0)$	$+(h'_0)$
		$\vdots$				$\vdots$	
		(a'_n)	$+(b'_n)$	$+\ \ldots$		$+(g'_n)$	$+(h'_n)$
						(a_0)	$+(b_0)$
						$\vdots$	$\vdots$
						(a_{m-1})	$+(b_{m-1})$
						(a'_0)	$+(b'_0)$
						$\vdots$	$\vdots$
						(a'_{q-2})	$+(b'_{q-2})$
							(a_0)
							$\vdots$
							(a_{m-1})
							(a'_0)
							$\vdots$
							(a'_{q-1})

If the general equation of degree $p+1$ be written in the form

$$(\alpha)^{p+1} + (\beta)^{p} + (\gamma)^{p-1} + \cdots \text{ etc.} = 0,$$

comparison of its derivates with linear functions of the given equations, carried so far as to include the terms of degree $k+1$ in these, at once suggests the possibility of a source $S=0$, in addition to those given in the table, where

$$S = (a_{m-1})^{p+1} + (b_{m-1})^{p} + \cdots + (h_{m-1})^{k+2} + (J_{m-1})^{k+1} + (K_{m-1})^{k}.$$

The J's in this cannot be put equal to the j's of the given equation; if they could, the source would be derived from the stationary set alone, contrary to the hypothesis that there are only $m-1$ equations of degree $p+1$ belonging to this set. (The impossibility of determining the J's in this manner arises algebraically from the fact that the j's are not subject to the linear relations that prevail for the a's, b's, etc.)

Since the two derivates of S must be expressible in terms of the given equations of degree p and their derivates, it is at once evident that $D_x S$ must agree with E^p_{m-1}, and that

$$D_y S \quad [= (a_m)^p + (b_m)^{p-1} + \cdots]$$

must agree with

$$\eta_1 E^p_0 + \eta_2 E^p_1 + \cdots + \eta_m E^p_{m-1},$$

each modified by linear multiples of the equations of degree k. Let J' be written for $\eta_1 j_0 + \eta_2 j_1 + \cdots + \eta_m j_{m-1}$, etc., then the equations to be satisfied are those derived from

$$(J_{m-1})^k \equiv (j_{m-1})^k + \sum \lambda (a_0)^k + \sum \lambda' (a'_0)^k,$$
$$(J_m)^k \equiv (J'_m)^k + \sum \mu (a_0)^k + \sum \mu' (a'_0)^k.$$

These are

$$\begin{aligned}
J_{m-1} &= j_{m-1} + \sum \lambda_1 a_0 + \sum \lambda'_1 a'_0,\\
J_m &= j_m + \sum \lambda_1 a_1 + \sum \lambda'_1 a_2 = J'_m + \sum \mu_1 a_0 + \sum \mu'_1 a'_0,\\
&\cdots\\
&\cdots\\
J_{m-1+k} &= j_{m-1+k} + \sum \lambda_1 a_k + \sum \lambda'_1 a'_k = J'_{m-1+k} + \sum \mu_1 a_{k-1} + \sum \mu'_1 a'_{k-1},\\
J_{m+k} &= J'_{m+k} + \sum \mu_1 a_k + \sum \mu'_1 a'_k.
\end{aligned}$$

Here there are m λ's and μ's, and q λ''s and μ''s, where $m+q=k-1$. The double values for $J_m, \cdots, J_{m-1+k}$ yield k equations of the type

$$\begin{aligned}
(\mu_1 - \eta_1\lambda_m) a_0 + (\mu_2 - \lambda_1 - \eta_2\lambda_m) a_1 + \cdots + (\mu_m - \lambda_{m-1} - \eta_m\lambda_m) a_{m-1}\\
+ \mu'_1 a'_0 + (\mu'_2 - \lambda'_1) a'_1 + \cdots + (\mu'_q - \lambda'_{q-1}) a'_{q-1} - \lambda'_q a'_q + J'_m - j_m = 0.
\end{aligned}$$

In these there are $m + q + 1$ ($= k$) combinations of the unknown quantities; and since the absolute terms in these k equations are not zero we obtain for each of these combinations a determinate value, in general different from zero. These leave the m λ's arbitrary, as also $\lambda'_1, \lambda'_2, \cdots, \lambda'_{q-1}$, the m μ's and $\mu'_2, \mu'_3, \cdots, \mu'_q$ being expressed in terms of these; but they assign to λ'_q and μ'_1 definite numerical values, which it will be shown immediately are in general different from zero. The m λ's and $q - 1$ λ''s appearing in the general values for $(J_{m-1})^{k+1}$ are non-significant; they simply allow for the modification of the source by multiples of the equations of degree $k + 1$; if these be taken to be zero, and L, M be written for λ'_q, μ'_1, the J's become

$$\begin{aligned} J_{m-1} &= j_{m-1} + La'_{q-1}, \\ J_{m+i} &= j_{m+i} + La'_{q+i} = J'_{m+i} + Ma_i \qquad (i = 0, \cdots, k-1), \\ J_{m+k} &= J'_{m+k} + Ma'_k. \end{aligned}$$

Hence the source is

$$S = (a_{m-1})^{p+1} + (b_{m-1})^p + \cdots + (h_{m-1})^{k+2} + (j_{m-1})^{k+1} + L(a'_{q-1})^{k+1} + (K)^k,$$

where the coefficients in $(K)^k$, equivalent to two independent ones, are to be determined by means of the two given curves. (The j_{m+k}, etc., denote definite numerical quantities which present themselves in the solution of the equations in such a manner that these names are the obvious ones for them.) The component $L(a'_{q-1})^{k+1}$ in this source indicates that it is derived from the two sets together: the vanishing of L would mean that the source belongs to the one set, which is contrary to the hypothesis. Thus it is seen that *the separation of the equations into sets at the foundation does not indicate any permanent cleavage; the sets are connected by this equation of degree $p + 1$*. It may be noted that

$$D_x S = D_y(E^{p+1}_{m-2} + L E'^{k+1}_{q-2}),$$

where E' indicates the equation $(a'_{q-2})^{k+1} + (b'_{q-2})^k = 0$ of the second set. This shows that we are in a position to continue the process of ascent.

While this is all that is necessary for the proof of the applicability of the theorem of ascent, the form of the complete result is of interest. I have not worked it out to the end, but a few steps make the general law perfectly plain. Let E^{p+t}_h be written for $(a_h)^{p+t} + (b_h)^{p+t-1} + \cdots + (j_h)^{k+t}$; the additional equation of degree $p + 1$ has been shown to differ from E^{p+1}_{m-1} at the terms of degree $k + 1$ by a multiple of E'^{k+1}_{q-1}; there are two extra equations of degree $p + 2$ which differ from E^{p+2}_{m-2} and E^{p+2}_{m-1} at the terms of degree $k + 2$ by multiples of E'^{k+2}_{q-2}, E'^{k+2}_{q-1} and expressions of lower degree; three extra equations of degree $p + 3$ which differ from E^{p+3}_{m-3}, E^{p+3}_{m-2} and E^{p+3}_{m-1} by multiples of E'^{k+3}_{q-3}, E'^{k+3}_{q-2}, E'^{k+3}_{q-1} and lower expressions, and so on, until finally the source of the second

set, $E_0^{(q+k)}$, makes its appearance. There are still m equations of the next higher degree, from whose $m+1$ derivates the source $E_0^{(q+k)}$ results by elimination of the highest terms, namely, those with coefficients a to h included. After this, the equations diminish in number by unity at every stage. (See § 10.)

The very slight modifications required when the second set becomes stationary at or before the degree k, in which case $m+q$ is equal to k instead of $k-1$, do not interfere with this conclusion; nor does the breaking up of the stationary set further—the different sets can be compounded in turn.

24. It has now been shown that the equations obtained by the process of ascent do not break away into sets; they form one system, and lead to one equation of some finite degree p, which is the prime equation for the base-point determined by the two curves. As all the equations due to the nature of the curves at the origin are included in the system, it follows that the base-point determined by the two curves is a one-set point. We have now proved the

THEOREM: *two curves satisfy precisely one prime equation,*
which is the first theorem of § (2).

If two or more sources are given for two curves, then by means of this theorem it is seen that they and their derivates can be exhibited as component parts of a more extensive system, arising from the one prime equation that is satisfied by the two curves; and if an expression E and all its derivates vanish, then E is either the prime equation itself, or one of its derivates. The system derived from E is either a part or the whole of the one-set system.

25. *Proof of Theorem* (2).—The results already obtained enable us to find the equations satisfied by more than two curves. Suppose that $m+1$ equations of degree p have been found satisfied by $h+2$ curves, and that the process of ascent for two of the curves yields m equations of degree p; the complete system satisfied by the two is

$$\lambda_1 E_1 + \lambda_2 E_2 + \cdots + \lambda_m E_m = 0.$$

Imposing on the λ's the conditions afforded by the remaining h curves, namely,

$$\lambda_1 E_1^{(i)} + \lambda_2 E_2^{(i)} + \cdots + \lambda_m E_m^{(i)} = 0 \qquad (i = 1, \cdots, h).$$

we shall obtain the system satisfied by the $h+2$ curves. If $h \leqq m$, this is

$$\begin{vmatrix} E_1 & E_2 & \cdots & E_h & \lambda_{h+1} E_{h+1} + \cdots + \lambda_m E_m \\ E_1^{(1)} & E_2^{(1)} & \cdots & E_h^{(1)} & \lambda_{h+1} E_{h+1}^{(1)} + \cdots + \lambda_m E_m^{(1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ E_1^{(h)} & E_2^{(h)} & \cdots & E_h^{(h)} & \lambda_{h+1} E_{h+1}^{(h)} + \cdots + \lambda_m E_m^{(h)} \end{vmatrix} = 0.$$

Hence if $m > h$, there are $m - h$ independent equations of degree $p+1$; that is, if the number of equations is not less than the number of curves, we can ascend to a smaller number.

But this smaller number will not suffice unless its derivates, whose number cannot exceed $2(m-h)$, yield all the $m+1$ equations. We must have therefore,

$$2(m-h) \geqq m+1, \quad \text{that is,} \quad m-h \geqq h+1.$$

The test is, therefore, that *the number of equations of degree* $p+1$ *shall be not less than* $h+1$.

If the equations of degree $p+1$ are not so many as $m+1$, they do not form the complete system, and a certain number of those of degree p must be given independently. Inasmuch as there are at most $2(m-h)$ equations of degree p derived from those of degree $p+1$, there must be given independently at least $m+1-2(m-h)$ of degree p; hence the system of equations contains at least $m-h+m+1-2(m-h)$, that is, $h+1$. Thus with $h+2$ curves to attend to, we can ascend to $h+1$ equations, but no further; and we have the

Theorem: $t+1$ *independent curves satisfy precisely* t *prime equations*, which is the second theorem of § 2.

To find the equations for given curves, the process of ascent seems simplest. It is most conveniently applied to two of the curves in the first place, after which linear functions of the equations obtained must be chosen so as to be satisfied by the remaining curves.

III. *The number of intersections of two curves at the origin.*

26. The proof that the number of points of intersections of two curves at a common point is the same as the number of equations contained in the one-set system proper to the point depends on a particular arrangement of the equations. It must be noticed in the first place that in the prime equation no coefficients z_h^p of the highest degree need be considered to be absent, for this can be obviated by a change of axes if necessary.

In the diagram in which the x-derivates are arranged in vertical lines, namely,

$$\begin{array}{lll} E & & \\ E_0^1 & E_1^1 & \\ E_0^2 & E_1^2 & E_2^2 \\ \vdots & \vdots & \ddots \end{array}$$

let E_1^1 be replaced by $E_1^1 + M_1 E_0^1 + M_2 E_0^2 + \cdots$, the multipliers M being chosen so as to remove every z_0; thus E_1^1 as modified contains terms

$$(z_1^{p-1} + \cdots) + (z_1^{p-2} + \cdots) + \cdots.$$

When all the members of the column are treated in a similar manner, the same modification is produced; moreover, the relation of each member to the one above remains unaltered, it is still the x-derivate. Similarly, let E_2^2 be replaced by an expression containing no terms z_0, z_1, and so on. If it should happen that not only every z_0, but also every z_1, disappears from E_1^1, this must be taken as the head of the third column, and its y-derivate as head of the second column, with similar modifications in the arrangement if other z's disappear. If *every* z^h disappears when any E of degree h is modified with a view to eliminating a particular z^h, the resulting equation is of lower degree, but this does not affect the arrangement of columns. Since the equations of degree $k-1$ reduce to

$$z_0^{k-1}=0\,,\; z_1^{k-1}=0\,,\; \cdots,\; z_{k-1}^{k-1}=0\,,$$

it follows that every one of the first k suffixes is represented by a column; the diagram now consists of k columns, not necessarily arranged according to height, whose bases are the equations

$$(z_0)^{k-1}=0\,,\; (z_1)^{k-1}=0\,,\; (z_2)^{k-1}=0\,,\; \cdots,\; (z_{k-1})^{k-1}=0\,,$$

where $(z_h)^{k-1}$ is written for a linear function of z^{k-1}'s, with no suffix below h. There are also to be taken into account the equations expressing that every z with an index $< k-1$ vanishes; when these are arranged below the others, according to their suffixes, the diagram represents all the equations belonging to the base-point. The numbers of equations in the columns are

$$p_0+1\,,\; p_1,\; p_2-1\,,\; \cdots,\; p_s-(s-1),\; \cdots p_{k-1}-(k-2),$$

where the degree of the leading equation of any column is denoted by p with the proper suffix.

Figs. (1) and (2) illustrate possible forms with $p_0=10\,,\, k=5\,.$

degree 10	*				
	*	*			
	*	*	*		
	*	*	*		
	*	*	*	*	
	*	*	*	*	*
degree 4	*	*	*	*	*
	*	*	*	*	
	*	*	*		
	*	*			
	*				

FIG. 1.

degree 10	*				
	*				*
	*			*	*
	*		*	*	*
	*	*	*	*	*
	*	*	*	*	*
degree 4	*	*	*	*	*
	*	*	*	*	
	*	*	*		
	*	*			
	*				

FIG. 2.

27. If an equation and all its derivates are satisfied, it is known that the equation is a member of the one-set system. For the recognition of this, however, it is not necessary that the fact be stated for all the derivates. *If an expression E and its first k y-derivates $E_1^1, E_2^2, \cdots, E_k^k$, as well as all the x-derivates of these, vanish for two curves that have at the origin a multiple point of order k, then all the derivates of E vanish, so that E is a member of the one-set system proper to the point.* The proof of this depends on showing that the equations are not independent. If at every degree the x-derivates of the $k+1$ equations $E, E_1^1, E_2^2, \cdots, E_k^k$ are independent, they lead to $k+1$ equations of degree k, which determine the order of the multiple point as $k+1$ instead of k. Hence the x-derivates are not independent at every degree. But if a linear relation connects the x-derivates of degree h, and the equations of degree $h+1$ are combined as indicated by this same linear relation, the only z's remaining will be $z_{h+1}^{h-1}, z_h^h, z_{h-1}^{h-1}$, etc. This gives a relation in coefficients of powers of y only, which can be avoided by taking axes with no specialized relation to the curve. Hence we see that the $k+1$ equations of degree p, where p is equal to the degree of E_k^k, cannot be independent; they must reduce to k by means of a linear relation. If now this relation does not involve E_k^k, let it involve no E^k beyond E_h^k, then

$$E_h^k = (E_0^k,\ E_1^k,\ \cdots,\ E_{h-1}^k)^1.$$

But since $E_h^k = D_x E_h^{k-1}$, this may be written

$$D_x E_h^{k-1} = D_x(E_0^{k-1},\ E_1^{k-1},\ \cdots,\ E_{h-1}^{k-1})^1;$$

this shows that from $E_0^{k-1}, E_1^{k-1}, \cdots, E_h^{k-1}$ all z's except $z_{p+1}^{p+1}, z_p^p, \cdots$, etc. can be eliminated, a particular case which as we have seen can be avoided. Consequently the relation connecting the derivates of degree p does involve E_k^k; it gives therefore

$$E_k^k = \text{a linear function of } E_0^k, \cdots, E_{k-1}^k.$$

Hence

$$\begin{aligned} D_y E_k^k &= \text{a linear function of the } y\text{-derivates of } E_0^k, \cdots, E_{k-1}^k, \\ &= \text{a linear function of the } x\text{-derivates of } E_1^k, \cdots, E_k^k, \\ &= 0, \text{ by hypothesis.} \end{aligned}$$

Since the x-derivates of $E_0^k, E_1^k, \cdots, E_k^k$ are given equal to zero, we now know that all the derivates of degree $p-1$ are zero. The same argument applies to the next line of derivates, and so on. Hence the given expression E and *all* its derivates vanish, so that the system E forms a part, or even the whole, of the one-set system proper to the base-point determined by the two curves.

28. *Proof of Theorem* (3).—To determine the number of intersections at the origin of two curves u, v of orders l, m, where

$$u = a_0^k x^k + a_1^k x^{k-1} y + a_2^k x^{k-2} y^2 + \cdots + a_0^{k+1} x^{k+1} + \cdots,$$
$$v = b_0^k x^k + b_1^k x^{k-1} y + b_2^k x^{k-2} y^2 + \cdots + b_0^{k+1} x^{k+1} + \cdots,$$

let the equation of either curve w, arranged by powers of y, be

$$w = w_0 + y w_1 + y^2 w_2 + \cdots = 0,$$

where

$$\begin{array}{lll}
w_0 & = & z_0^k x^k + z_0^{k+1} x^{k+1} + \cdots, \\
w_1 & = & z_1^k x^{k-1} + z_1^{k+1} x^k + z_1^{k+2} x^{k+1} + \cdots, \\
w_2 & = & z_2^k x^{k-2} + z_2^{k+1} x^{k-1} + z_2^{k+2} x^k + z_2^{k+3} x^{k+1} + \cdots, \\
\cdot & \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
w_k & = & z_k^k + z_k^{k+1} x + \cdots + z_k^{2k-2} x^{k-2} + z_k^{2k-1} x^{k-1} + z_k^{2k} x^k + z_k^{2k+1} x^{k+1} + \cdots, \\
w_{k+1} & = & z_{k+1}^{k+1} + z_{k+1}^{k+2} x + \cdots + z_{k+1}^{2k-1} x^{k-2} + z_{k+1}^{2k} x^{k-1} + z_{k+1}^{2k+1} x^k + z_{k+1}^{2k+2} x^{k+1} + \cdots,
\end{array}$$

etc.

Then any u_h is w_h with a written for z, and any v_h is w_h with b written for z. Also $w_{h-1} = D_y w_h$.

Let y be eliminated from the equations $u = 0$, $v = 0$, then the lm values of x corresponding to the points of intersection are to be found by solving the equation

$$\left|\begin{array}{ccccccc}
u_0 & u_1 & u_2 & \cdots & u_l & & \\
 & u_0 & u_1 & & u_{l-1} & u_l & \\
 & & u_0 & m \text{ rows} & u_{l-2} & u_{l-1} & u_l \\
 & & \cdots & & & & \\
 & & \cdots & & & & \\
v_0 & v_1 & v_2 & & & & \\
 & v_0 & v_1 & & & v_m & \\
 & & v_0 & l \text{ rows} & & v_{m-1} & v_m
\end{array}\right| = 0.$$

It is to be determined how many zero roots are present, that is, what power of x is a factor in this determinant. We shall find that the elements of each row can be combined so as to eliminate the lower powers of x only with the help of the equations of the one-set system.

Any expression linear in the z's with any one index, such as

$$\lambda_0^p z_0^p + \lambda_1^p z_1^p + \cdots + \lambda_p^p z_p^p,$$

is the coefficient of x^p in

$$\lambda_0^p w_0 + \lambda_1^p x w_1 + \cdots + \lambda_p^p x^p w_p;$$

if however $\lambda_0^p = 0$, this set of terms is, more simply, the coefficient of x^{p-1} in

$$\lambda_1^p w_1 + \lambda_2^p x w_2 + \cdots + \lambda_p^p x^{p-1} w_p.$$

In general,

$$\lambda_s^p z_s^p + \lambda_{s-1}^p z_{s-1}^p + \cdots$$

is the coefficient of x^{p-s} in

$$\lambda_s^p w_s + \lambda_s^{p+1} x w_{s+1} + \cdots.$$

Hence the terms given by a more general expression E_s^p, namely,

$$\lambda_s^p z_s^p + \lambda_{s+1}^p z_{s+1}^p + \cdots + \lambda_s^{p-1} z_s^{p-1} + \lambda_{s+1}^{p-1} z_{s+1}^{p-1} + \cdots + \cdots,$$

form the coefficient of x^{p-s} in

$$\lambda_s^p w_s + \lambda_{s+1}^p x w_{s+1} + \lambda_{s-2}^p x^2 w_{s+2} + \cdots + x\{\lambda_s^{p-1} w_s + \lambda_{s-1}^{p-1} x w_{s+1} + \cdots\}$$

$$+ x^2\{\lambda_s^{p-2} w_s + \lambda_{s+1}^{p-2} x w_{s+1} + \cdots\} \cdots,$$

that is, in

$$(\lambda_s^p + x\lambda_s^{p-1} + x^2\lambda_s^{p-2} + \cdots) w_s + (x\lambda_{s+1}^p + x^2\lambda_{s+1}^{p-1} + \cdots) w_{s+1} + \cdots\cdots,$$

or, say, in $S = X_s w_s + X_{s+1} w_{s+1} + \cdots\cdots$.

The coefficient of the next lower power of x in this expression S is obtained by taking one step to the eft in every w; this replaces z_q^p by z_q^{p-1}, that is, by its x-derivate. Hence the coefficient of $x^{p-s-1} = D_x \cdot$ coefficient of x^{p-s}, and so on. If then $E_s^p = 0$ is an equation of the one-set system, not only is the coefficient of x^{p-s} in S zero, but also the coefficient of every lower power of x. Hence x^{p-s+1} is the lowest power of x that is present in this combination of w's. If now the columns of the determinant are multiplied by the expressions thus indicated, namely, the $(s+1)$-th by X_s (which, it is to be noted, does not contain x as a factor), the $(s+2)$-th by X_{s+1}, etc., the $(s+1)$-th column can be replaced by

$$\begin{array}{l}
X_s u_s \quad + X_{s+1} u_{s+1} + X_{s+2} u_{s+2} + \cdots \\
X_s u_{s-1} + X_{s+1} u_s \quad + X_{s+2} u_{s+1} + \cdots \\
X_s u_{s-2} + X_{s+1} u_{s-1} + X_{s+2} u_s \quad + \cdots \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
X_s v_s \quad + X_{s+1} v_{s+1} + X_{s+2} v_{s+2} + \cdots \\
X_s v_{s-1} + X_{s+1} v_s \quad + X_{s+2} v_{s+1} + \cdots \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{array}$$

As regards the u-rows and v-rows separately, each row is the y-derivate of the row immediately above it; hence if the coefficient of x vanishes in the first row, it vanishes in all following rows, that is, the lowest power of x that is present in the first row is a factor in every member of the column. Thus by means of the equation of degree p that heads the $(s+1)$-th column of the particular arrangement of the equations given in § 26 we can make x^{p-s+1} a factor in the $(s+1)$-th column of the determinant, and this is the highest power of x that can be obtained as a factor in this column in this manner. Since X_s does not contain x as a factor, no irrelevant powers of x have been introduced in the process. Consequently we obtain, in the first k columns of the determinant, powers of x with exponents $p-(s-1)$, that is, with exponents

$$p_0+1, \quad p_1, \quad p_2-1, \quad \cdots, \quad p_{k-1}-(k-2);$$

x presents itself therefore as a factor with the exponent

$$(p_0+1)+p_1+(p_2-1)+\cdots+(p_{s-1}-\overline{s-2})+\cdots+(p_{k-1}-\overline{k-2}),$$

which is the total number of equations in the system. Hence *the number of intersections is at least as great as the number of equations in the one-set system.*

29. It has still to be shown that no higher power of x is present as a factor. Returning to the original determinant, let the columns be combined so as to replace the $(s+1)$-th by

$$\begin{array}{l} X_s u_s \quad + X_{s+1} u_{s+1} + \cdots \\ X_s u_{s-1} + X_{s+1} u_s \quad + \cdots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ X_s v_s \quad + X_{s+1} v_{s+1} + \cdots \\ X_s v_{s-1} + X_{s+1} v_s \quad + \cdots \end{array}$$

where the X's are expressions in x, about which nothing is known except that X_s does not contain x as a factor. Then if x^{t+1} is a factor in every member of the column, it is a factor in the determinant. In every member of the column the coefficient of x^t is zero, as also the coefficient of every lower power of x. Now forming the expression $X_s w_s + X_{s+1} w_{s+1} + \cdots$, and denoting the coefficient of x^t in this by E, we see that the coefficient of x^{t-1} is $D_x E$, and so on; we see moreover that the coefficient of x^t in $X_s w_{s-1} + X_{s+1} w_s + \cdots$ is $D_y E$, and so on. If then x^{t+1} is a factor in the first $k+1$ u-rows, and in the first $k+1$ v-rows, it follows that for the two curves u, v the expression E, with k successive y-derivates, and all their x-derivates, are zero. Consequently, by the theorem of § 27, E is a member of the system proper to the base-point determined by these two curves. This argument does not apply, however, if the expression

E has no x-derivates; that is, if E involves only $z_k^k, z_{k+1}^{k+1}, z_{k+2}^{k+2}, \cdots$. In this case the combination of w's obviously begins with w_k or a later one; to make even a single x a factor in such a combination, it is necessary that there be values of $\lambda_1, \lambda_2, \cdots$ satisfying the equations

$$\begin{aligned} z_k^k + \lambda_1 z_{k+1}^{k+1} + \lambda_2 z_{k+2}^{k+2} + \cdots &= 0, \\ \lambda_1 z_k^k \quad + \lambda_2 z_{k+1}^{k+1} + \cdots &= 0, \\ \lambda_2 z_k^k \quad + \cdots &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \end{aligned} \tag{I}$$

for both u and v. In other words, there must be quantities $\lambda_1, \lambda_2, \cdots$ that satisfy the m equations obtained from these by writing a for z, and the l equations obtained by writing b for z.

Again, writing $x = 0$ in the equations $u = 0$, $v = 0$, and dividing by y^k, we obtain the points other than the origin where these two curves meet the axis of y, by means of the two equations

$$a_k^k + a_{k+1}^{k+1} y + a_{k+2}^{k+2} y^2 + \cdots + a_l^l y^{l-k} = 0,$$
$$b_k^k + b_{k+1}^{k+1} y + b_{k+2}^{k+2} y^2 + \cdots + b_m^m y^{m-k} = 0.$$

The condition that these two equations have a common root is the vanishing of the determinant

$$\begin{vmatrix} a_k^k & a_{k+1}^{k+1} & a_{k+2}^{k+2} & a_l^l & & \\ & a_k^k & a_{k+1}^{k+1} & a_{l-1}^{l-1} & a_l^l & \\ \cdot & \cdot & \cdot\ m-k \text{ rows} & & & \\ b_k^k & b_{k+1}^{k+1} & b_{k+2}^{k+2} & & b_m^m & \\ & b_k^k & b_{k+1}^{k+1} & & b_{m-1}^{m-1} & b_m^m \\ \cdot & \cdot & \cdot\ l-k \text{ rows} & & & \end{vmatrix},$$

which is precisely the condition that the $m - k$ equations

$$\begin{aligned} a_k^k + \lambda_1 a_{k+1}^{k+1} + \lambda_2 a_{k+2}^{k+2} + \cdots &= 0, \\ \lambda_1 a_k^k \quad + \lambda_2 a_{k+1}^{k+1} + \cdots &= 0, \end{aligned}$$
etc.

and the $l - k$ equations (II)

$$\begin{aligned} b_k^k + \lambda_1 b_{k+1}^{k+1} + \lambda_2 b_{k+2}^{k+2} + \cdots &= 0, \\ \lambda_1 b_k^k \quad + \lambda_1 b_{k+2}^{k+2} + \cdots &= 0, \end{aligned}$$
etc.

be consistent. But these equations (II) are a part of equations (I); if then equations (I) hold, that is, if x is a factor in the combination, equations (II) show that this is due to the fact that the two curves have a point of intersection on the axis of y, but distinct from the origin. This depends on a specialized choice of axes, and can be avoided. It is the only way in which a power of x can present itself as a factor without the help of the equations of the one-set system; hence the number of intersections at the origin is that found in § 28, which completes the proof of the

THEOREM: *the number of intersections of two curves at the origin is equal to the number of equations in the one-set system.*

30. In applying the theory to particular curves, one point worthy of notice is that the degree of the prime equation may be higher than the order of either curve by which it is determined. This is natural, inasmuch as the equations relate not only to the given curves u, v, but to every curve of the system $Xu + Yv$. From another point of view, the degree of the prime equation simply tells to what order of small quantities the branches through the origin are specified, and this has no connection with the order of the curve. The equations, regarded as imposing conditions on curves of order n, $n < p$, are subject to mutilation, since all z's for which the index is greater than n are now zero. But these missing terms must be supplied in any general application of the theory. For instance, the cubics

$$y - x^2 - y^2 + x^2y - xy^2 + y^3 = 0,$$
$$y - x^2 + xy - x^3 - xy^2 = 0,$$

have nine-point contact at the origin. The prime equation, of degree 8, and its x-derivates, are

$$E^8 = z_0^8 + z_1^7 + z_2^6 + z_3^5 + z_1^4 - z_1^3 + 2z_2^3 + z_1^2 - 2z_2^2 + z_1^1 = 0,$$
$$E^7 = z_0^7 + z_1^6 + z_2^5 + z_3^4 + z_1^3 - z_1^2 + 2z_2^2 + z_1^1 = 0,$$
$$E^6 = z_1^5 + z_0^6 + z_2^4 + z_3^3 + z_1^2 - z_1^1 = 0,$$
$$E^5 = z_0^5 + z_1^4 + z_2^3 + z_1^1 = 0,$$
$$E^4 = z_0^4 + z_1^3 + z_2^2 = 0,$$
$$E^3 = z_0^3 + z_1^2 = 0,$$
$$E^2 = z_0^2 + z_1^1 = 0,$$
$$E^1 = z_0^1 = 0,$$
$$E^0 = z_0^0 = 0.$$

The y-derivates are linear functions of these; hence the single column of equations gives the complete set. Now as applied to the cubic these equations reduce to

$$E^8 = -z_1^3 + 2z_2^3 + z_1^2 - 2z_2^2 + z_1^1 = 0,$$

$$E^7 = z_1^3 - z_1^2 + 2z_2^2 + z_1^1 = 0,$$

$$E^6 = z_3^3 + z_1^2 - z_1^1 = 0,$$

$$E^5 = z_2^3 + z_1^1 = 0,$$

$$E^4 = z_1^3 + z_2^2 = 0,$$

etc.,

and these mutilated equations are equivalent to eight only, on account of the relation $E^8 + E^7 - E^5 = 0$, which does not hold for the proper equations. Nevertheless the one-set system of equations contains nine members, even when applied to the cubic; strictly the order of the curve is irrelevant, since the equations relate to the system $Xu + Yv$, for which this reduction in number does not take place.

31. A remark in Dr. MACAULAY's second paper may properly be noted here. He points out that the equations can be arranged in such an order that stopping at any point, we have the equations of a base point. All that is necessary for this is that no one of the equations shall appear before any of its own derivates. If we break off at any point in the series we thus have a number of sources with all their derivates; that is, the equations proper to a certain t-set point, contained in the given one-set point.

Conclusion.

32. In conclusion, a few remarks of a general character may not be out of place. The fundamental idea of the theory, namely, that the equations can be dealt with as derivates of a comparatively small number, seems to be of real importance. In these pages I have dwelt on certain aspects of the question, hoping to attract other minds to it; for while I believe that the somewhat lengthy presentation here made may possibly be much simplified, yet I confess I cannot see in what direction. The ideas involved are simple and direct, even if their development be somewhat tedious. It seems that more may be done by the direct discussion of the equations, here attempted, than by the ingenious but artificial processes of the original memoirs.

One question as to which there is scope for investigation relates to the geometrical interpretation, not only of the individual equations (which is to some extent answered in § 31), but of the diagram by which the whole set is repre-

sented. It has been shown that the equations arranged as in § 10 present one of the following appearances

$$
(1)\quad \begin{array}{llllll}
\ast \\
\ast & \ast \\
\ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast
\end{array}
\qquad
(2)\quad \begin{array}{llllll}
\ast \\
\ast & \ast \\
\ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast
\end{array}
\qquad
(3)\quad \begin{array}{llllllll}
\ast \\
\ast & \ast \\
\ast & \ast & \ast \\
\ast & \ast & \ast \\
\ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast
\end{array}
$$

(3′)

where (3) may have any number of steps, each accounted for as in § 10. Figures (1) and (2) are easily interpreted. If the curves have each a k-point, with no contacts, the scheme of equations is of the simple form (1); the prime equation is of degree $2(k-1)$. If one curve has a k-point, and the other an h-point, where $h > k$, and there is no contact (the simple case when NOETHER's theorem is in question) the diagram is of the form (2). This may also be looked upon as representing two curves each with a k-point, and with contact of a certain kind. The prime equation is of degree $h+k-2$. This set gives for the number of intersections the sum of k terms

$$h+k-1,\quad h+k-3,\quad h+k-5,\quad \cdots,\quad \text{that is, } hk. \quad \text{(See § 26.)}$$

As to (3), most probably the vertical boundaries on the right indicate contacts of some of the branches of the two curves, the number of branches involved being shown by the breadth of the section of the figure. This point appears to be well worth investigation; it looks very much as though the form of the diagram might turn out to be a complete indication of the relation of the two curves to one another. Whether the idea will be of any use in the investigation of compound singularities appears doubtful, since the conditions for superlinear branches do not involve the coefficients linearly, and one would hesitate to undertake the discussion of the theory even of *quadratic* prime and derived equations.

BRYN MAWR COLLEGE, PENNSYLVANIA.

ON THE CIRCUITS OF PLANE CURVES*

BY

CHARLOTTE ANGAS SCOTT

1. The nature of the individual circuits (or complete branches) that make up a curve of order n has not received very much attention. Von Staudt (1847) distinguished between odd and even circuits; Moebius (1852, *Ueber die Grundformen der Linien der dritter Ordnung*) by projection on to a sphere from the centre, brought out even more clearly the distinction, since the odd circuit is represented on the sphere by a single line, the even circuit by two distinct lines. Cayley (1865, *On Quartic Curves*, Collected Papers, vol. 5, op. 361) applied the Moebius projection to the quartic, and thus proved that not only the individual circuits of any non-singular quartic, but all the circuits at once, can be projected into the finite part of the plane. He pointed out that this conclusion does not hold as regards the non-singular sextic; there exists such a curve, composed of a single circuit, which cannot be projected into the finite part of the plane. On a sphere, this circuit is not confined to one hemisphere. In the concluding paragraph the remark is made that a quartic with one node may consist of two odd circuits; such a quartic is not the projection of any finite curve. Clifford (1870, *Synthetic Proof of Miquel's Theorem*) mentioned another sextic that is not confined to one hemisphere. Zeuthen, in his classical discussion of quartic curves (1874, Mathematische Annalen, vol. 7) emphasized the distinction between odd and even circuits, and proved the important theorem that in one, and only one, of the two regions † into which a plane is divided by a non-singular even circuit, odd circuits can lie. He showed (p. 426) that a quartic can be composed of a single even circuit, cutting itself twice, of such a form that every straight line meets it in at least two real points. There may be an additional double point of any kind, or the curve may have a simple circuit, necessarily a simple oval. A quartic of this character, with $p = 0$, can be obtained by quadric inversion from a conic which separates one of the three fundamental points from the other two. If such a circuit be projected on to the sphere, it will be found that the two lines by which it is represented interlace.

2. The theorems mentioned suggest that there is a general theorem as to the existence of circuits that cannot be projected into the finite part of the plane,

* Presented to the Society April 26, 1902. Received for publication May 29, 1902.

† On a sphere, there are three regions.

that is, circuits some of whose intersections with every straight line are real. Let the maximum number of points in which a circuit is met by a straight line be called its order; let the minimum number be called the index of the circuit; it is obvious that

$$\text{order} - \text{index} \equiv 0 \pmod 2,$$

and that

$$\text{order} - \text{index} \geqq 2.$$

The general theorem can be stated in the form:

For every order n there exist curves, $p = 0$ *or* 1, *formed of a single circuit of index* $n - 2$.

The curves $p = 1$ may have also another circuit, without point- or line-singularities.

Further, for every order n there exist curves, $p = 0$ or 1, formed of a single circuit of index $n - 2r$, where r may have any value from 1 to $n/2$ if n be even, from 1 to $(n-1)/2$ if n be odd.

3. As a preliminary, in order to form some idea of the appearance of such a curve, consider a curve of order n ($p = n - 2$) composed of $n - 2$ odd circuits through O, namely

$$(x^2 + y^2)\prod_{i=1}^{i=n-2} l_i + \prod_{i=1}^{i=n-2} t_i = 0,$$

where the l's and t's represent real lines through O, alternating. Let these odd circuits be drawn so that they do not all pass through O, as in Fig. 1, for which $n = 5$. If now the first circuit be linked with the second, and the second with the third, as indicated in Fig. 1′, a single circuit of index 3 is formed with the help of two new double points B, C.

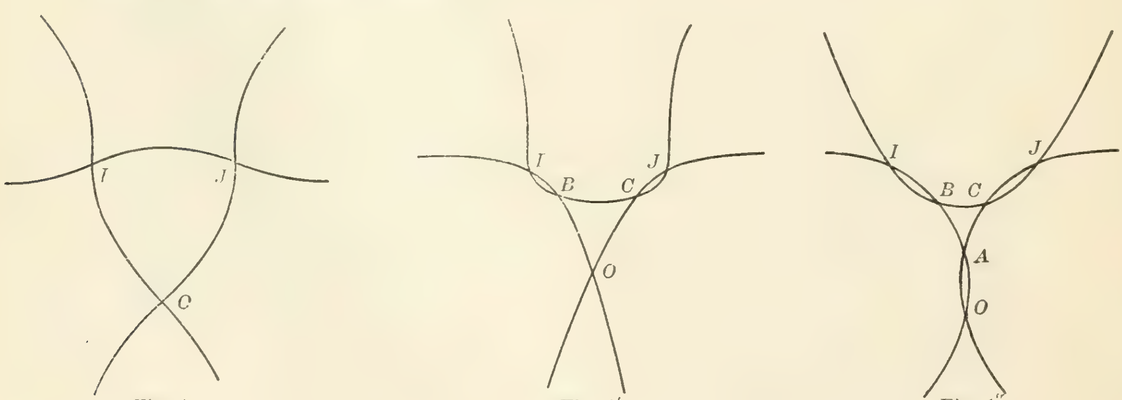


Fig. 1 *Fig. 1′* *Fig. 1″*

Similarly the $n - 2$ circuits, linked by means of $n - 3$ new double points, become a single circuit of index $n - 2$. If this process of deformation is

admissible, the resulting curve is a C_m, $p = 1$, composed of a single circuit, for which

$$\text{order} - \text{index} = 2.$$

A different linking of the circuits is shown in Fig. 1″, accomplished by means of three new double points; the curve is a quintic, $p = 0$.

4. The rigorous proof of the theorem is obtained by means of Cremona transformations. By this means also the validity of this process of linking the circuits is established.

When an oval is subjected to quadric inversion, its position with respect to the fundamental triangle OIJ determines the appearance of the resulting circuit. If the oval does not meet any of the fundamental lines, it transforms into an oval, with or without indentations; if it crosses any of the fundamental lines, but in such a manner as not to enclose any of the fundamental points, it becomesa branch cutting itself, but still such that it can be projected into the finite part of the plane. If however the oval surrounds one of the fundamental points, O, it necessarily meets in at least two real points all conics through OIJ, and since these correspond to the straight lines of the transformed plane, the resulting circuit is of index 2, of the type of the Zeuthen quartic circuit. A similar result follows if the oval surrounds two of the fundamental points.

An oval that passes through one of the fundamental points transforms into an odd circuit, of index 1, passing through the two fundamental points whose correspondents are the two fundamental lines that meet in the point through which the oval passes.

The quintic of which the existence was suggested by the process of deformation can actually be obtained by inversion from a quartic. Let $u = 0$ be a quartic consisting of three ovals; let O, I, J be points on these, chosen for the

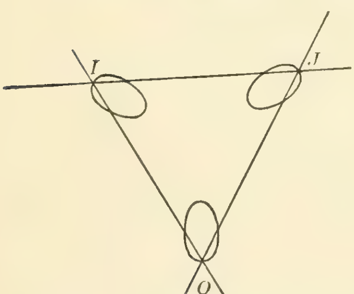


Fig. 2

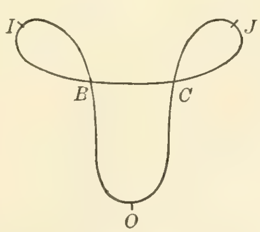


Fig. 2′

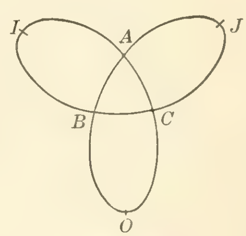


Fig. 2″

sake of simplicity so that the quartic touches a conic $v = 0$ at these points (Fig. 2). The quartics $u + \lambda v^2 = 0$, where λ is a variable parameter, form a system through OIJ, lying outside u if the sign of λ is properly chosen. These invert

into a system of quintics, composed of three odd circuits intersecting at O, I, J. If u, v are chosen so that the whole figure has a single axis of symmetry through O, the two double points B and C (Fig. 2′) present themselves for the same value of λ, and the circuits are linked as in Fig. 1′. If there is triangular symmetry, the three double points A, B, C appear for some one value of λ, (Fig. 2″), and the transformed curve is as shown in Fig. 1″.

In the general Cremona transformation the straight lines of the transformed plane are derived from rational curves passing through the fundamental points in the original plane, forming a Cremona net. If the curve to be transformed entirely surrounds a fundamental point of odd order, it is cut in at least two real points by every curve of the Cremona net; the transformed curve consequently meets every straight line in its plane in at least two real points, and is therefore of index 2. The number of fundamental points inside any one loop, link, or oval, makes no difference, so long as there are fundamental points of odd order outside also. Similarly the presence of a fundamental point of odd order on the boundary of a loop, link, or oval ensures one real intersection other than the fundamental point itself.

5. The simplest type of Cremona net is composed of curves of order q with a fixed $(q-1)$-point and $2(q-1)$ fixed simple points. By means of this net the theorem can be proved. It will be shown in the first place that for every order m there exists a curve with a multiple point of order $m-3$ and $2(m-3)$ links, if $p=1$, but with $2(m-3)+1$ links if $p=0$. Such curves are familiar among quartics, as shown in Fig. 3 ($p=0$) and Fig. 3′ ($p=1$).

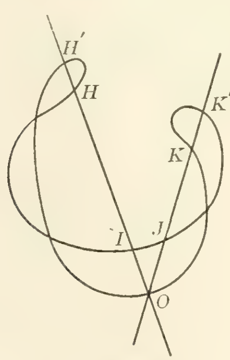


Fig. 3

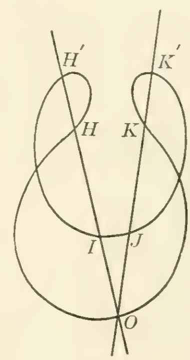


Fig. 3′

A point O is chosen on the part of the curve indicated in the figure, and from O two lines are drawn to meet the final loops of the curve in real points, which is certainly possible for these quartics; these lines meet the quartic again at I, J. By means of a quadric inversion with OIJ as fundamental points the curve is transformed into a quintic, with a double point at O since the line IJ meets

the quartic again in two points. The pairs of points H, H'; K, K' give rise to new double points at I, J, and the resulting quintic is of the desired type. (Figs. 4, 4.′)

Two lines can be drawn through O to meet the loops in real points, H, H', K, K'; these lines meet the base of the curve each in one point, necessarily real, and therefore available as fundamental points I, J. The line IJ meets the quintic again in three points, hence the transformed curve, a sextic,

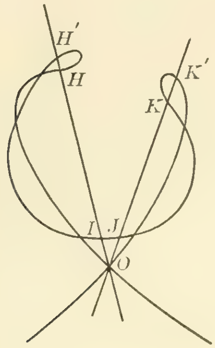


Fig. 4

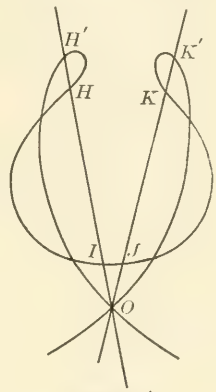


Fig. 4′

has a triple point at O; and as before the points HH', KK' give rise to double points at I, J, and the two series of links in the curve are increased each by unity. The curve is a C_6, with a 3-point, and $2.3 + 1$ links if $p = 0$, 2.3 links if $p = 1$.

In precisely the same manner we obtain from a C_m with an $(m - 3)$-point and $2(m - 3) + 1$ [or $2(m - 3)$] links a curve of order $2m - (m - 3) - 2$, that is, $m + 1$, with, at O, a point of order $m - 1 - 1$, that is, $m + 1 - 3$, and $2(m - 3) + 1 + 2$, that is $2\{(m + 1) - 3\} + 1$ [or $2(m - 3) + 2$, that is, $2\{(m + 1) - 3\}$] links. It is not necessary to consider whether the branches that form the multiple point at O are all real; this is immaterial. What is essential is that the links be all real, and this is shown by the manner in which the curve is produced.

6. We now apply a Cremona transformation of the kind mentioned, choosing q so that there shall be a fundamental point available for the interior of every link. This requires $q = m - 2$. The curve C_m with an $(m - 3)$-point and $2(m - 3)$ links is transformed by means of curves of order $m - 2$ with a fixed $(m - 3)$-point and $2(m - 3)$ fixed simple points. The transformed curve is therefore of order

$$n = m(m - 2) - (m - 3)^2 = 4m - 9,$$

and the index, to which every link contributes 2, is at least $2 \times 2(m - 3)$.

Since however $4(m-3) = 4m - 9 - 3$, which gives for *order* — *index* the odd value 3, the index must be greater than $4(m-3)$. It cannot be greater than $4m - 9 - 2$, hence

$$\text{index} = 4m - 11 = n - 2.$$

This proves the theorem for every n of the form $4m - 9$, that is, $4k - 1$; for the intermediate values all that is necessary is to place $1, 2, 3$ of the fundamental points on the boundaries of links instead of inside. Every such placing of a fundamental point diminishes both order and index by unity, and thus leaves undisturbed the relation, now proved for all values of n,

$$\text{order} - \text{index} = 2.$$

The fundamental curves for the Cremona net here used are

(i) $2(m-3)$ straight lines joining O to the other fundamental points, singly;

(ii) a C_{m-3} with an $(m-4)$-point at O, passing singly through the other fundamental points.

Since each of the straight lines (i) meets C_m in 3 points that do not lie at the fixed points, the C_n has $2(m-3)$ triple points if $n = 4m - 9$, but $1, 2, 3$ of these are replaced by double points if $n = 4m - 9$ diminished by $1, 2, 3$. The fundamental C_{m-3} gives rise to a multiple point of order $4(m-3)$ on C_n, diminished by $1, 2, 3$ for the special cases named; the order of this point is therefore in all cases $n - 3$. There are also the unaltered double points of the original curve. These make up exactly the right number for the C_n with $p = 0$ or 1.

If now h of the fundamental points are placed outside the links instead of inside, the order is not affected, but the index is diminished by $2h$. In this way, for the curve of order $4m - 9$ the index can be made as low as 1, instead of $4m - 11$. For the intervening orders the lowest index, 1 or 0, is obtained by placing the last $1, 2, 3$ fundamental points on the base of the curve.

This completes the proof of the theorem.

7. For greater values of p it can be shown that the index can attain to certain specified values, but proof is still lacking that these are the highest possible values * for any given n and p. If p be of the form $4t-2, 4t-1, 4t, 4t+1$, then for every order n, where $n \geqq 2(t+1)$, there exists a curve composed of a single circuit for which

$$\text{order} - \text{index} = 2(t+1).$$

For the proof of this, the first step is to demonstrate the existence of curves of every order m of the type shown in Figs. 5 and 5′, each with an $(m-3)$-point at O. For any such curve

* In fact, they are not the highest values, as witness the sextic, $n = 6$, $p = 10$, index $= 2$.

$$p = \tfrac{1}{2}(m-1)(m-2) - \tfrac{1}{2}(m-3)(m-4)$$

diminished or not by unity; that is, $p = 2m - 6$, Fig. 5, and $p = 2m - 5$, Fig. 5′.

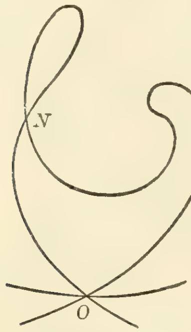


Fig. 5

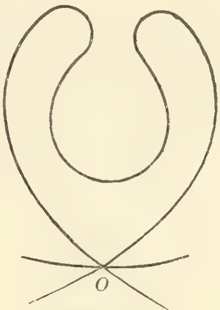


Fig. 5′

If the existence of such a curve, $u_m = 0$, be assumed for any value of m, it follows at once for the value $m + 2$.

(1) Let m be even, the desired C_{m+2} is given by

$$f_2 u_m = \lambda F_m \phi_2,$$

with a sufficiently small value for λ, where f_2 is an imaginary line-pair through O, F_m a set of m imaginary lines through O, and ϕ_2 a pair of imaginary lines, not through O, but intersecting at N if this other node exists.

(2) Let m be odd; then with the same interpretation for f, F, and ϕ the conditions are fulfilled by

$$f_2 u_m = \lambda F_{m-1} z \phi_2,$$

where $z = 0$ is the straight line at infinity.

This proves that if curves of the kind indicated exist for order m, they exist for order $m + 2$. They do exist for $m = 4$; and the equation

$$y u_4 = \lambda F_2 z \phi_2$$

proves that they do exist for $m = 5$, and consequently for every value of m. Any such curve may have also certainly one oval or even two, possibly more.

If we now transform these curves by quadric inversion, exactly as for the case of $p = 0$ or 1, we obtain from each a series of curves, as shown in Fig. 6, where curves of the same order are represented in the columns, and curves with the same p in the rows.*

* These diagrams show the nodes and links accurately as to number and general situation; they are not intended to show directions to infinity or inflexions. In fact, some inflexions not actually existing on the curves have been inserted on the links, for clearness in drawing.

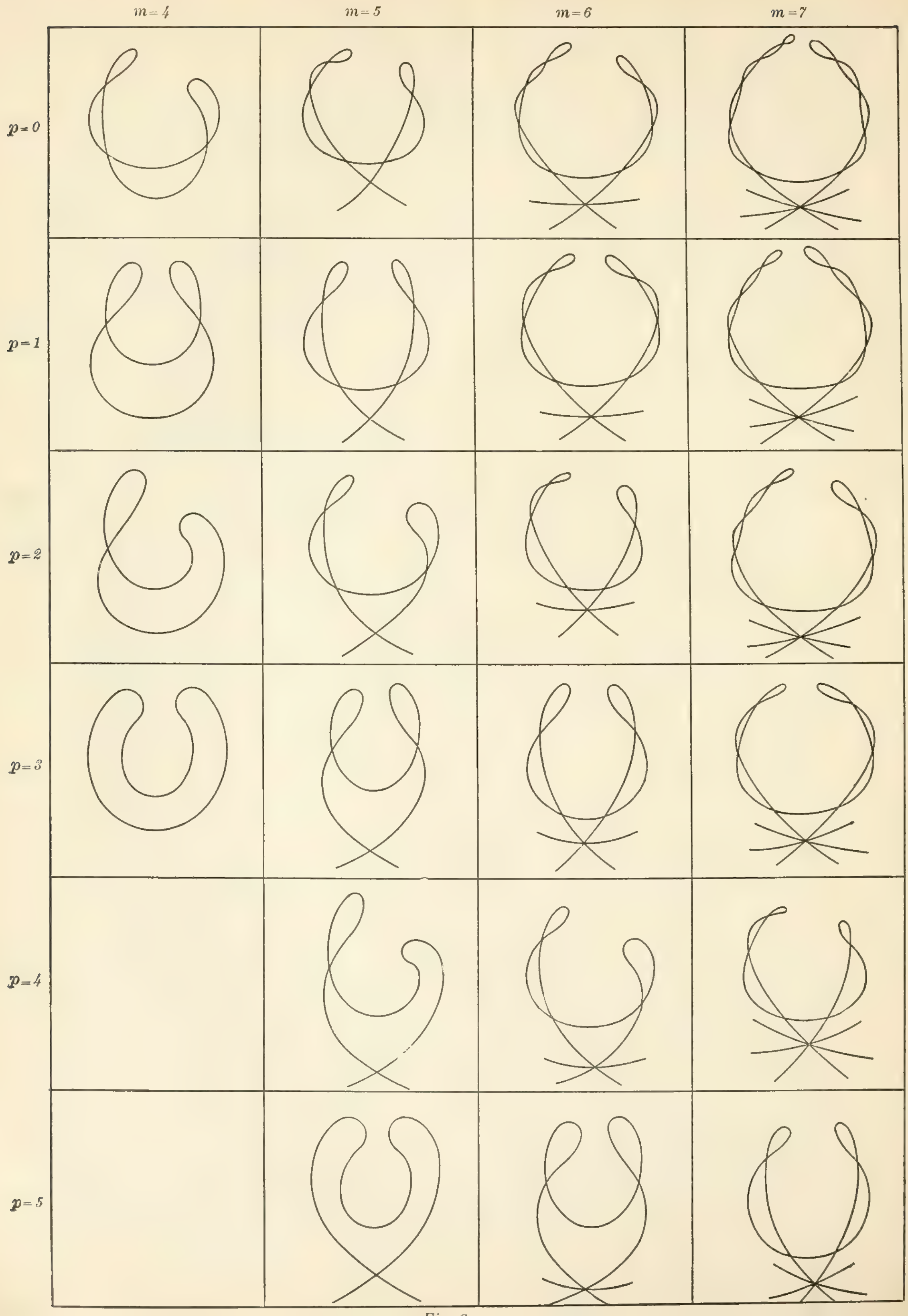


Fig. 6.

Any such curve is a C_m with an $(m-3)$-point and q links, where $q = 2m - 5 - p$. This is to be subjected to a Cremona transformation by means of curves of order $m-k$, with a fixed $(m-k-1)$-point and $2(m-k-1)$ simple points, where $2(m-k-1) \geqq 2m-5-p$, in order that there may be a fundamental point available for every link. Hence $2k \leqq p+3$. The fundamental points left over, in number $p+3-2k$, can be placed on the base of the C_m; their effect is then to diminish the order of the transformed curve, for which we have the value

$$n = m(m-k) - (m-3)(m-k-1) - (p+3-2k) = 4m - k - p - 6.$$

Exactly as before,

$$\text{index} \geqq 2(2m-5-p),$$

and therefore

$$\text{index} \geqq n - (p+4-k).$$

The index is therefore greatest, for a given n and p, when k is as great as possible, that is, when

$$2k = p+3 \quad \text{or} \quad p+2.$$

(1) If p be even, $2k = p+2$;

$$\text{index} \geqq n - \tfrac{1}{2}(2p+8-p-2) \geqq n - \tfrac{1}{2}(p+6);$$

hence

$$\text{order} - \text{index} \leqq \tfrac{1}{2}(p+6).$$

If $\frac{1}{2}(p+6)$ be even, that is, if $p = 4t-2$,

$$\text{order} - \text{index} = 2(t+1).$$

If $\frac{1}{2}(p+6)$ be odd, that is, if $p = 4t$,

$$\text{order} - \text{index} \leqq 2t+3,$$

hence

$$\text{order} - \text{index} = 2(t+1).$$

(2) Similarly if p be odd, it is either of the form $4t-1$ or of the form $4t+1$ and in both cases,

$$\text{order} - \text{index} = 2(t+1).$$

Smaller values for the index, with or without a corresponding diminution in the order, can of course be obtained as before by placing some of the fundamental points on the boundaries of links, or outside the links, instead of inside.

8. In the construction of one of these complicated circuits by the linking of simpler circuits the Zeuthen circuit, derived by inversion from a simple oval, presents itself as fundamental, ranking with the simple oval and the simplest

type of odd circuit. Two nodes are essential, and four real inflexions (Fig. 7), unless there is a cusp or another node (Fig. 7′), which obviates the necessity for two of the four inflexions. For this circuit the name "double-odd circuit" has been suggested.* Topologically, it is derivable by deformation from two odd circuits, the six real inflexions being reduced to the four actually existing by means of the extra node due to the linking of the circuits.

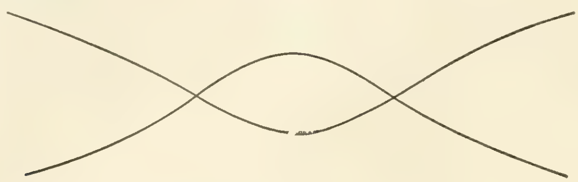


Fig. 7

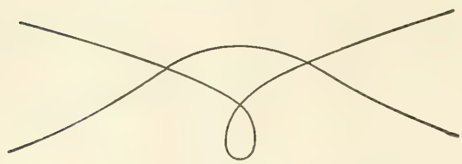


Fig. 7′

If $u = 0$ be one of the curves of order m with an $(m-3)$-point and q links, of which k are occupied by fundamental points in order to give rise to a circuit of index $2k$ or $2k+1$, then $u = \lambda$, where λ is a small constant of proper sign, contains at least q ovals, of which k are occupied by fundamental points. By the Cremona transformation each of these k ovals gives rise to a double-odd circuit, and the single complicated circuit, $u = 0$, arises from the linking of these k double-odd circuits with the circuit derived from the base of the curve. Thus the circuit of index $2k+1$ is derived from k double-odd circuits and one other; or, we may say, it is derived (topologically) from $2k+1$ odd circuits. When $1, 2, 3$ fundamental points are placed on boundaries, $1, 2, 3$ of the double-odd circuits are replaced by simple odd circuits. In any case, the index of any curve here considered is equal to the number of the odd circuits from which the curve can be derived, topologically, by the process of linking.

Since each odd circuit has three real inflexions, a fact proved by MOEBIUS, the circuit of index i, produced from i odd circuits with the help of $i-1$ new double points if $p = 1$, has at least $3i - 2(i-1)$, that is, $i+2$ real inflexions. In particular, the curve for which $i = n-2$ has at least n real inflexions if $p = 1$, and at least $n-2$ if $p = 0$. KLEIN's equation connecting the numbers of the real singularities, namely,

* R. GENTRY, 1896, *On the Forms of Plane Quartic Curves.*

order + no. real inflexions + 2 no. isolated tangents

= class + no. real cusps + 2 no. isolated points,

becomes in this case (since there cannot be an isolated tangent, which would require four imaginary intersections with a straight line)

$$\begin{aligned} \text{order} + \text{no. real inflexions} &= \text{class} \\ &= 2n\,, \text{ if } p = 1\,, \\ &= 2n - 2\,, \text{ if } p = 0\,. \end{aligned}$$

Hence

$$\begin{aligned} \text{no. real inflexions} &= n\,, \text{ if } p = 1\,, \\ &= n - 2\,, \text{ if } p = 0\,. \end{aligned}$$

If however $p = 0$ in virtue of an isolated point instead of an extra node on the circuit itself, the number of real inflexions $= n$.

9. Any odd circuit in the transformed plane meets the fundamental straight lines each in an odd number of points, and therefore corresponds to a circuit passing an odd number of times through each of the simple fundamental points in the original plane. Now this circuit, passing in and out of the links, has at least $2k$ real intersections with the C_m. Hence not only every straight line, but also *every odd circuit*, cuts the transformed curve in $n - 2$ (in $n - 2r$) real points. This indicates an essential difference between the circuits here considered and the non-singular sextic circuit of index 2, since, as has already been mentioned, there are odd circuits that do not meet this sextic. It seems therefore that it may be necessary to introduce some such term as *circulation*, to denote the number of odd circuits from which a given one can be derived by deformation, or, perhaps more conveniently, the minimum number of times that the circuit must be crossed in passing from an arbitrary point on the sphere to the opposite point, when the path is entirely at our disposal. This aspect of the case I hope to consider at some future time; it is sufficient at present to remark that obviously

$$\text{index} - \text{circulation} \equiv 0 \quad (\text{mod } 2),$$

and that for the complicated circuits here considered

$$\text{index} = \text{circulation}\,.$$

Bryn Mawr College, Pennsylvania.

NOTE ON THE

REAL INFLEXIONS OF PLANE CURVES*

BY

CHARLOTTE ANGAS SCOTT

The determination of the possible situation of the real inflexions of a curve $u = 0$ is often practically very difficult. The work can sometimes be simplified by means of the fact that real inflexions lie only in the region for which $u_{11}u_{22} - u_{12}^2$ is zero or negative, where u_{11}, u_{22}, u_{12} are written for $\partial^2 u/\partial x^2$, $\partial^2 u/\partial y^2$, $\partial^2 u/\partial x\partial y$. This expression, G, depends on the relation of the curve $u = 0$ to the line $z = 0$: a corresponding expression can of course be obtained with reference to any line, and for a real inflexion every such expression must either be zero or have a negative value.

The proof is extremely simple. The polar conic of any point is

$$x^2 u_{11} + y^2 u_{22} + z^2 u_{33} + 2yzu_{23} + 2zxu_{31} + 2xyu_{12} = 0;$$

for a double point or a point of inflexion this represents a pair of straight lines, namely, the tangents at the double point, or the inflexional tangent together with another line which does not pass through the point of inflexion. This line-pair will be imaginary if any one such function as $u_{11}u_{22} - u_{12}^2$ is positive; it will be real if no one of these functions is positive, that is, if every one is either zero or negative. This is ensured if any one such function is negative, or if two are zero.

The expression G is regularly used in discriminating between real and imaginary tangents at a node, but it does not appear to have been used in considering the possibility of real inflexions. If we equate it to zero, we obtain a curve $G = 0$, derived from u and the line z, the locus of points whose polar conics meet z in coincident points. If z is the line at infinity, G is the locus of points whose polar conics are parabolas, including parallel straight lines and coincident straight lines, in which last case the points are cusps or higher singularities.† The curve G divides the plane into regions; the polar conics of points in the

* Presented to the Society December 28, 1901. Received for publication August 16, 1902.

† The higher singularities here referred to are double points with coincident tangents; multiple points of order higher than 2 lie on $G = 0$ for a different reason, namely, on account of the vanishing of u_{11}, u_{22}, etc.

region in which the expression G is positive meet the line z in imaginary points (ellipses and imaginary straight lines) and the polar conics of points in the region defined by G negative meet z in real points (hyperbolas and real straight lines).

It would seem that the curve thus derived from u and a line, as the locus of points whose polar conics touch the line, or as the envelope of the so-called second polars of points on the line, merits consideration. The name "diacritic" suggests itself; G is the diacritic (or diacritic curve) of z with respect to u. If u be a cubic, G is simply the "Poloconik" of the line.

The diacritic of z is the same with respect to all curves of the family $u = k$ ($u = kz^n$ in homogeneous coördinates); it separates the region in which isolated points can lie from the region in which the other nodes (and all real inflexions) are to be found, and thus it must plainly be of importance in the consideration of the critic centres of this pencil. These critic centres are the points common to the first polars of points at infinity, $u_1 + \lambda u_2 = 0$; for this linear system they take the place of an envelope for a more general system, while the diacritic is the envelope of the second polars of points at infinity, $u_{11} + 2\lambda u_{12} + \lambda^2 u_{22} = 0$.

The diacritic is the locus of points of contact of curves $u_1 =$ constant with curves $u_2 =$ constant. This fact is also expressed in the more general statement that the increment of u_1 along the curve $u_2 = 0$ depends at any point on the sign of G (since $\delta u_1 = u_{11}\delta x + u_{12}\delta y$, where $u_{12}\delta x + u_{22}\delta y = \delta u_2 = 0$), and consequently changes sign only when $G = 0$. This shows that in passing along $u_2 = 0$ from one critic centre to the next we pass over G; in general the critic centres are alternately isolated points and nodes with real tangents.

BRYN MAWR COLLEGE, PENNSYLVANIA.

Reprinted from THE ASTROPHYSICAL JOURNAL, Vol. XIV, No. 2, September, 1901.

THE

ASTROPHYSICAL JOURNAL

AN INTERNATIONAL REVIEW OF SPECTROSCOPY
AND ASTRONOMICAL PHYSICS

VOLUME XIV SEPTEMBER, 1901 NUMBER 2

THE ABSORPTION SPECTRUM OF CHLORINE.

By ELIZABETH R. LAIRD.

HISTORICAL INTRODUCTION.

IN 1833 Brewster[1] discovered that the spectrum of white light which has passed through a layer of nitrogen peroxide is crossed by a number of black lines. It was natural that physicists should at once begin to look for similar phenomena in the case of other gases, and the colored ones received particular attention. W. H. Miller and Daniell[2] found such absorption lines with iodine, bromine and euchlorine (a mixture of chlorine and its oxides). The lines produced by iodine and bromine appeared to be equidistant and of equal intensity. Using pure chlorine they failed to find anything but a general absorption in the violet. W. A. Miller[3] investigated a number of gases in this connection, and made rough drawings of the absorption spectra of iodine, bromine, nitrogen peroxide, chlorine dioxide, and "perchloride of manganese;" but he also failed to obtain a line absorption due to chlorine. His apparatus, however, consisted only of a

[1] *Phil. Mag.* [3], **2**, 360, 1833; **8**, 384, 1836.

[2] *Pogg. Ann.*, **28**, 386, 1833. [3] *Phil. Mag.* [3], **27**, 81, 1845.

telescope and a single flint-glass prism placed twenty feet from a slit admitting diffused daylight; and it is not certain that he used a column of more than nine inches of chlorine. Later, in 1859, M. E. Robiquet[1] stated that there was not the least appearance of lines in the spectrum of chlorine even when the light had passed through 4.5 meters of the gas, but that the spectrum consisted of a general illumination in the green and yellow. His source of light was an incandescent platinum wire, and he used a single-prism spectroscope. Many other attempts were made to discover a chlorine line absorption spectrum, as the known similarities between this gas and bromine and iodine led to the belief that such a spectrum existed; but all these attempts failed on account of the lack of suitable apparatus. Finally, in 1869, Morren,[2] using a spectroscope of fine flint-glass prisms and a length of two meters of chlorine, succeeded in obtaining the desired spectrum. He gives a general description of its appearance. According to his observations the lines begin above the *b* solar lines and extend below F to about $\lambda 4820$. Beyond $\lambda 4750$ the sunlight is totally absorbed. The lines differ in intensity, width and grouping, and are placed irregularly. He mentions having made a drawing, but it was apparently not published. Later, Gernez[3] unaware of the previous work of Morren, rediscovered the lines. He tried first a length of 1.5 meters of chlorine at atmospheric pressure, but he obtained the lines more distinctly by using a tube of chlorine three times as long. His spectroscope had two prisms, and a Drummond lamp was the source of light. He states that this gives a spectrum extending into the "violet," hat the lines begin a little below the D lines and extend to the "violet," which is totally absorbed. He remarks also that the lines are unlike those of bromine and iodine in having variable intensity and irregular grouping.

Liveing and Dewar[4] have investigated the general absorption of chlorine in the ultra-violet. As a source of light they used

[1] *C. R.*, **49**, 606, 1859.

[2] *Pogg. Ann.*, **137**, 165, 1869; *C. R.* **68**, 376, 1869.

[3] *C. R.*, **74**, 660, 1872.

[4] *Chem. News*, **47**, 121, 1883.

the spark of an induction coil between iron electrodes. The tubes used had quartz ends and the dispersing prism was of quartz. They found that a small quantity of chlorine gave one absorption band stretching from $\lambda 3560$–$\lambda 3020$. As the quantity of chlorine was increased the band widened, and the following different measurements of its width are given — $\lambda 3968$–$\lambda 2755$; $\lambda 4415$–$\lambda 2665$; $\lambda 4650$–$\lambda 2630$; with the greatest amount of chlorine used they found that the absorption stopped at $\lambda 2550$.

The absorptive power of chlorine for the long heat rays was investigated by Tyndall.[1] He found that, excepting air, oxygen, nitrogen and hydrogen, it absorbed less than any other gas on which he experimented. K. Ångström and W. Palmaer[2] found that the infra-red spectrum consisted of a single band, which with an Argand lamp as source of radiation, and a column of 11.8 cm of chlorine, extended from $\lambda 32300$ to $\lambda 60700$ (Å. U.), the maximum absorption being at $\lambda 42800$.

Liquid chlorine is said by Gänge[3] to absorb the extreme red down to about $\lambda 6970$ or $\lambda 6860$; from there on the light is transmitted to about $\lambda 5120$, from which point absorption begins again, and is complete at $\lambda 5030$.

The emission spectrum of chlorine has been investigated at different times. Plücker[4] was the first to obtain it, but in his first experiments the spectrum did not last long enough for purposes of drawing; later, together with Hittorf,[5] he studied its spectrum as obtained from Geissler tubes and made a careful drawing of it. Van der Willigen[6] and Salet[7] studied the spectrum of the gas at atmospheric pressure. The former used the spark from an induction coil, the latter used a Holtz machine;

[1] TYNDALL, *Contributions to Molecular Physics in the Domain of Radiant Heat*, p. 80.

[2] *Öfversigt af K. Vet. Akad. Förhandl.*, No. 6, 389, 1893.

[3] GÄNGE, *Lehrbuch der angewandten Optik in der Chemie*, p. 217.

[4] *Pogg. Ann.*, **106**, 83, 1858.

[5] *Phil. Trans.*, **155**, 24, 1865.

[6] *Pogg. Ann.*, **106**, 624, 1859.

[7] *Ann. de Chim. et de Phys.* [4], **28**, 24, 1873.

both give drawings of the spectrum obtained. Ångström[1] and Hasselberg[2] measured a few lines. The latter remarked them in using vacuum tubes the glass walls of which contained chlorine. Lecoq de Boisbaudran[3] and Eugen Demarçay[4] identified as belonging to chlorine some lines obtained from the spark discharge between a platinum wire and hydrochloric acid; both give measurements on these lines. Ciamician[5] studied the relation of the spectrum of chlorine to the spectra of iodine and bromine; also the variation in the aspect of the chlorine lines for different pressures. The latest and most accurate measurements are by Eder and Valenta,[6] who used a Rowland grating and photographed almost the entire spectrum, finding about four hundred lines. The majority of these lines are in the ultra-violet or violet, but they extend through the blue and green into the yellow; and some, which Eder and Valenta observed but did not measure, are in the red. Those in the violet and ultra-violet are characterized as sharper than those in the green or yellow, the latter being, for the most part, broad or indistinct.

Investigations on the absorption spectrum of chlorine other than those already mentioned, do not appear to have been made, and in no case are measurements of the wave-lengths given. In a recent paper J. Koenigsberger,[7] referring to the effect of temperature on the line absorption of gases, speaks as if Brewster had investigated the absorption spectrum of chlorine, but he gives no exact reference. I have made careful search, but have been unable to find any work of Brewster's on the subject.

The study of the line absorption spectra of gases has

[1] *C. R.*, **73**, 369. [2] *Bull. de l'Acad. de St. Petersb.*, **28**, 405, 1881.

[3] LECOQ DE BOISBAUDRAN, *Spectres lumineux*, 1874.

[4] EUGEN DEMARÇAY, *Spectres lumineux*, 1895.

[5] *Sitzungsb. d. kais. Akad. d. Wiss.*, Wien, **77**, 839, 1878; **78**, 4 Abth., 1878.

[6] *Denksehr. d. kais. Akad. d. Wiss.*, Wien, **68**, 1899.

[7] *Ann. der Phys.* [4], **4**, 806, 1901.

received, on the whole, very little attention since the use of gratings has made possible more exact measurements. Considerable work has been done on the band absorption of liquids including the effect of temperature, etc. The influence of single solvents and of mixtures of solvents has also been investigated. There are some measurements on the absorption of vapors, but the apparatus used has been of low dispersive power, and the measurements are accordingly not very exact. The absorption of the various constituents of the air has been investigated and compared with the telluric lines of the solar spectrum, but few independent measurements of the wave-lengths have been made. From the reversal of emission lines, the absorption of hot vapors has been deduced. The work of Hasselberg[1] on the absorption of iodine and bromine is still the main example of an attempt to measure the lines in absorption spectra of gases under normal conditions as exactly as is done for emission spectra. This is is the more to be regretted as it is certain that a study of the line absorption spectra of gases in connection with emission spectra at low temperatures must increase our knowledge of the processes involved in both the emission and absorption of light, and aid us ultimately in picturing the constitution of the molecules. A further investigation of the absorption spectrum of chlorine seemed, therefore, not without interest, and the following pages give the details of some experiments on this subject.

APPARATUS.

The physical laboratory at Bryn Mawr College is provided with a 15 ft. Rowland grating of 15,000 lines to the inch and a ruled surface of 11×4 cm, which is mounted after the usual Rowland method. The camera holds plates 30.5×3.5 cm, and is provided with the arrangement used by Rowland[2] for comparison spectra.

Plates made by C. S. Schleussner, of Frankfurt a. M., were

[1] *Mem. de l'acad. de St. Petersb.*, 7, 36, No. 17, 1889; *K. Svenska Vet. akad. Handl.* No. 3, 24, 1891.

[2] *Phil. Mag.* [5], **27**, 378.

used to photograph in the blue and violet, and Seed's orthochromatic plates or Cramer's isochromatic plates were used in the green and yellow. The developer used throughout the investigation was mixed after the formula given by L. E. Jewell.[1]

In the green and yellow of the first order for slit width about 0.001 in. the time of exposure for the source alone, whether Sun or electric arc, was from six to ten minutes; with the tube of chlorine interposed it was from twenty to thirty minutes. In the second order and with slit width about 0.002 in. the exposures were about three times those given above.

On account of the large number of standard lines which it provides, the Sun is the most desirable source of light for this kind of work; but as the solar beam after entering the grating room from the heliostat traverses a distance of only 34 cm before falling on the slit, it was questionable whether this would admit a sufficiently long column of chlorine to produce the desired spectrum. The attempts of earlier investigators with columns of this length failed to show any traces of absorption lines; but as the apparatus at their disposal was so different from that here used it was thought worth while to make a trial. The result showed that this length of column was insufficient; there were suggestions of lines, but the absorption was too weak to make certain of the existence of lines without further confirmation.

A tube 65 cm long was then used with the arc as the source of light. Decided absorption lines appeared; but as increasing the column of chlorine promised to intensify the lines, a tube 1.37 meters long, the longest which could be placed between the condensing lens and the slit, was finally used. This tube was of glass of 5.2 cm internal diameter and was closed with plane glass plates. Metal caps screwed on to metal collars, which were cemented on to the outside of the tube, held the glass plates in

[1] ASTROPHYSICAL JOURNAL, **11**, 242, 1900.

place. Holes bored in the glass tube corresponding to similar ones in the collars, to which nipples were fitted, served as inlet and outlet for the chlorine; later, it was found better to allow the metal collars to project beyond the tube in order to avoid boring the glass, as the cementing on of the collars seemed to strain the glass to such an extent that boring a hole in it almost invariably caused cracking.

An attempt was made to use this tube with the Sun as source of light by placing it between the slit and the grating; with large slit with 0.0007 in., a fairly good photograph was obtainable, but diminishing the slit-width increased the indistinctness of the lines until they became broad and fuzzy. This, doubtless, was due to the larger proportion of light reflected from the sides of the tube. Blackening the tube with lampblack did not overcome the difficulty sufficiently to make this method feasible.

Another tube for use with solar light was made of steel, 33.2 cm long and 1.6 cm internal diameter. It also was provided with caps which screwed over the ends and held on the glass plates. It was hoped that this would hold the gas under more than atmospheric pressure, and that the effect of a longer tube would be thus obtained. However, the first form of end and valve leaked under pressure, as no form of packing could be found which chlorine would not attack. The steel itself was attacked, as was expected, but the ferric chloride soon formed a layer over the surface and protected the rest in great measure; and since it is a solid its presence cannot invalidate the results. When, however, particles of the chloride fell on the glass ends they became very troublesome and necessitated the removal of the caps in order to clean the glass. This proceeding was attended with difficulties as a leak of chlorine through the cap stuck tube and cap together. After some ineffectual attempts to obviate these difficulties the scheme was abandoned for the time being.

It was then determined to utilize the space between the total

reflecting prism and the condensing lens for the solar beam, which latter lens is in a metal tube projecting outside from the wall of the room. A glass tube 80 cm long and 5.3 cm internal diameter was made for use in this position. Distinct absorption lines were thus obtained; those towards the less refrangible end were still faint. Photographs were taken of all parts of the visible chlorine absorption spectrum from the D lines down, using both arc and Sun as the source of light.

To obtain the ultra-violet portion the glass ends were removed from the last-mentioned tube and were replaced by lead ones in which were inserted oblong quartz plates put in with soluble glass or soft wax. These were 2.5×4.0 cm, the largest possessed by the laboratory. For the extreme portion the Sun could not be used as a source of light, as the heliostat mirror at present in use is not of speculum metal, but of silvered glass, and the more refrangible rays are already absorbed. The arc was therefore used with a system of two quartz lenses.

The chlorine for these experiments was at first prepared from sulphuric acid, sodium chloride, and manganese dioxide in the usual way; that used later was taken from a cylinder of liquid chlorine, supplied by Messrs. Eimer and Amend, of New York, passed through washing and drying bottles and then led into the tubes. This cylinder gave a most convenient supply at any desired time, and the spectrum obtained with the chlorine from this source corresponded exactly with that obtained from the gas prepared by myself.

Further tests were made, however, for impurities which might cause errors in the results. As such, chlorine oxides were especially to be feared; but very careful tests showed that no trace of chlorine oxides was present. Carbon dioxide and a small fraction of air were found, the two making up in an extreme case one fourth of the total volume. Carbon dioxide has, however, no absorption in the visible part of the spectrum. P. Ballei[1] states that 75 meters of this gas under pressures up to

[1] *Nuovo Cimento*, **9**, 172, 1899.

twenty atmospheres showed no trace of an absorption spectrum; thus its presence, mixed in with the chlorine, but forming no chemical compound with it, can have no effect except to change the total pressure. The same is true for the small amount of air present.

For assistance in making the analysis of the chlorine, my thanks are due to Dr. Kohler, professor of chemistry at Bryn Mawr College.

MEASUREMENTS.

The pitch of the screw of the dividing engine with which the plates were measured is approximately 1 mm; the head reads to 0.005 mm and the vernier to 0.001 mm. The errors of turn and of run of the screw were investigated before final measurements were made.

To determine a possible error in turn the distance along a given line between two fine lines drawn with a diamond point on steel, less than a tenth of a millimeter apart, was measured, starting from different positions of the head. For this purpose a microscope of magnifying power 35 was used. The variation in the readings was found to be not greater than the possible error of setting. The greatest deviation from the mean was 0.0015 mm and the average deviation was less than one half of this amount, or not more than 0.002 Å. U., when turned into wave-lengths. As the lines to be measured were not sufficiently sharply defined to attain to greater accuracy than this, no correction was applied.

The error in turn was examined over 15 cm of the screw, and found similarly to be negligible.

The microscope used for measuring the lines on the photographic plates had a magnifying power of 16. The wave-lengths were computed in the usual way from those of standard lines. The wave-lengths of all reference lines used were taken from Rowland's table of standard wave-lengths. On the arc plates there was some difficulty in obtaining standards, but sufficient were always found from which to calculate the reduction

factor; other Rowland lines were sometimes used for comparison.

Both Sun and arc plates had their disadvantages. Even with the comparison solar spectrum photographed beside the solar plus the chlorine absorption spectrum, it is difficult to decide at times whether a line belongs to chlorine or not. If a line is so close to a solar line as to be inseparable from it, though its existence may be apparent from the broadening of the solar line, or from the increase of intensity, accurate setting is very difficult without some *a priori* knowledge of the width and character of the line. On the other hand, when the light from the glowing positive pole of the electric arc gives the spectrum the plates abound with black lines on a continuous dark background. The chlorine absorption lines appear, then, as white lines mixed in with black ones, and the danger arises of mistaking for an absorption line, the narrow space between two adjacent black lines, which is white merely by contrast. As a great number of photographs have been examined in this respect it is scarcely possible that any lines measured have been thus mistaken. There is the possibility that a chlorine line may fall on a black line, in which case it may not appear as an absorption line at all. Especially is this possible in a part of the green carbon band.

All of the lines given have been measured on arc plates, a large number have been measured also on Sun plates, as will be seen from the tables. As many of the lines are ill-defined, the second order plates did not offer much advantage over those of the first order, except for those lines which are broken up into doubles in the second order, and as the time of exposure in the second order was very much longer, the first order was used more extensively. In the first order 1 mm on a plate corresponds to 3.6 Å U., approximately.

The average mean error of measurement is about 0.015 Å. U. As stated above, the only exact measurements of this kind previously made are due to Hasselberg. He claims an accuracy

of 0.02 or 0.03 Å. U. for his measurements on the sharpest lines of the absorption spectrum of iodine, lines which he used as normals; and as it is to be inferred that the lines of the bromine spectrum are less well-defined than those of iodine, the general accuracy is probably not so great for the measurements on the bromine spectrum. The error, then, in the present measurements is almost certainly less than that in previous work on absorption spectra, and is also not greater than that of the general measurements on emission spectra.

Explanation of Table I.—The measurements are given in the following table. The first column contains the number of separate measurements on the line; an *s* after the number indicates that some of the measurements were made on Sun plates. The second column contains the wave-length in Ångström units; the third, the mean error; and the fourth the intensities, reckoned on the scale 0, 1, 2, . . . 10, and details as to the appearance of the lines. Intensity 1 is given to the faintest lines that are clearly visible; 0 to lines that are not so; and 10 to the strongest lines.

N signifies ill-defined; *s*, sharp; *b*, broad; *vb*, very broad; *d*, double; *d?*, possibly double. From λ 4799 to λ 5165 the numbers are given for chlorine at one atmosphere pressure. From λ 5165 to λ 5218 measurements and intensities are given for a meter of chlorine both at atmospheric pressure, and at a pressure of two and a half atmospheres; from λ 5218 on the measurements are for a pressure of two and a half atmospheres.

TABLE I.

n	Wave-length	Mean error	Intensity and character	n	Wave-length	Mean error	Intensity and character
3	4799.11	0.017	0 *N*	6 *s*	4833.83	0.015	2 *N d*
4	4799.951	.011	0	6 *s*	4834.263	.010	2
3	4800.95	.015	0	4 *s*	4834.729	.012	2
4	4802.066	.011	0	8 *s*	4835.205	.007	4
3	4802.759	.011	0	5 *s*	4835.79	.018	3 *N*
6	4803.322	.013	1	5 *s*	4836.495	.014	4 *N*
3	4804.560	.023	0	9 *s*	4837.073	.006	4 *d?*
2	4805.13	.037	0	4 *s*	4837.54	.016	1
4	4805.94	.029	0	7 *s*	4838.02	.016	7 *b*
5	4806.57	.014	2	4	4838.94	.012	3
5	4807.081	.011	2	6 *s*	4839.435	.013	2
2	4807.38	.003	0	4	4839.85	.018	2 *N*
5	4809.081	.012	2	3	4840.31	.006	1 *N*
3	4809.66	.012	1	5 *s*	4840.973	.012	10
4	4809.949	.021	1	3 *s*	4841.41	.012	1
4	4810.603	.014	3	3 *s*	4841.667	.004	2
8 *s*	4811.168	.006	4	2	4841.96	.043	2 *N*
5 *s*	4811.899	.013	3	3	4842.93	.026	3 *b*
4	4812.598	.010	2 *N*	2	4843.21	.011	2
5 *s*	4813.047	.013	2 *N*	3	4843.71	.010	2
4	4813.747	.003	3 *N b*	2	4844.10	.020	2 *N*
6 *s*	4814.393	.008	3 *N b*	5 *s*	4844.601	.012	2 *N*
4	4814.77	.030	0	7 *s*	4845.209	.008	5 *b*
8 *s*	4815.353	.008	2 *b*	5 *s*	4846.017	.009	3
7 *s*	4815.832	.014	2 *b*	8 *s*	4846.534	.005	6
3	4816.43	.019	0	3 *s*	4847.13	.004	0
3	4816.77	.003	0	7 *s*	4847.736	.010	3 *N*
2	4817.44	.016	1 *N*	3	4848.49	.022	2
4	4817.927	.009	4	5 *s*	4849.122	.006	3 *b*
3	4818.34	.026	1	4 *s*	4849.647	.006	1
4 *s*	4818.75	.021	2	9 *s*	4850.141	.007	3 *N*
6 *s*	4819.15	.028	1 *b N*	9 *s*	4850.605	.009	3 *N*
4 *s*	4820.059	.011	2 *N*	9 *s*	4851.062	.007	3 *N*
3	4820.45	.022	2 *N*	4	4851.637	.007	1 *N*
4	4821.08	.014	3 *N*	7 *s*	4852.158	.009	5
7 *s*	4821.955	.012	2 *b*	6 *s*	4852.532	.008	2
6 *s*	4822.652	.009	2 *b N d?*	7 *s*	4853.30	.011	3 *d*
2	4823.22	.022	2 *N*	4	4853.99	.015	1 *N*
3	4823.69	.018	2 *N*	9 *s*	4854.411	.008	4
3	4824.21	.010	2 *b N*	4	4855.070	.011	1 *N*
8 *s*	4824.735	.007	2	5	4855.595	.012	8 *s*
5 *s*	4825.888	.013	4 *vb*	4	4855.96	.010	1
8 *s*	4826.887	.011	4 *b*	4 *s*	4956.647	.001	2
5 *s*	4827.630	.008	4 *b d?*	6 *s*	4856.966	.016	2
6 *s*	4828.614	.012	4 *b*	3 *s*	4857.30	.023	1 *N*
4	4829.41	.018'	3	6 *s*	4857.867	.010	9 *s*
6 *s*	4830.068	.011	3 *b*	4 *s*	4858.39	.016	1
8 *s*	4830.418	.005	3 *b*	4 *s*	4858.763	.008	1
3	4831.315	.003	3 *vb d*	4	4859.173	.010	5 *b*
5 *s*	4832.057	.009	2	6 *s*	4859.761	.012	5 *b*
4 *s*	4832.67	.020	3 *N*	3	4860.37	.013	2 *d?*
3	4833.13	.036	1	3 *s*	4860.84	.021	2 *N*

TABLE I—*Continued.*

n	Wave-length	Mean error	Intensity and character	*n*	Wave-length	Mean error	Intensity and character
4	4861.472	0.005	6	4	4888.63	0.016	2 *N*
4	4861.985	.014	2 *N*	5	4888.966	.012	4 *b*
3	4862.34	.019	2 *N*	4 *s*	4889.851	.007	1
8 *s*	4863.126	.008	5 *b*	5 *s*	4890.23	.019	2 *N*
4	4863.36	.011	1	5 *s*	4890.503	.009	2 *N*
5	4864.127	.004	5 *b d ?*	3 *s*	4891.05	.031	0
6	4864.991	.011	5 *b d*	3	4891.42	.014	2 *N*
3	4865.78	.023	0	4	4891.829	.013	3 *s*
4 *s*	4866.118	.016	2	2	4892.20	.002	0
4	4866.625	.007	6	2	4892.40	.018	0
5 *s*	4867.331	.006	3 *b*	4 *s*	4892.801	.006	2
4	4867.796	.010	1 *b N*	3	4893.11	.033	1 *N*
6	4868.499	.007	10	3	4893.52	.026	2 *N*
4	4868.90	.011	1	8 *s*	4893.695	.010	3 *N*
4 *s*	4869.83	.022	4 *b N*	4 *s*	4894.20	.021	1 *N*
4	4870.398	.009	4 *b*	4 *s*	4894.49	.022	1 *N d*
5	4871.123	.006	7 *b*	3	4894.77	.043	0
3	4872.19	.037	2 *N*	8 *s*	4895.317	.008	4 *b*
4	4872.526	.008	3 *N*	3	4895.69	.017	0
4	4873.11	.018	2 *N*	9 *s*	4895.950	.011	4 *b N*
5	4873.60	.018	2 *N*	4	4896.35	.021	2 *N*
3	4874.088	.006	3 *b N*	4	4896.888	.007	2
4 *s*	4874.650	.010	4	4 *s*	4897.119	.009	3
3 *s*	4875.19	.024	2 *b*	8 *s*	4897.572	.011	4 *b*
4	4875.801	.016	4	3	4897.75	.017	3 *N*
4	4876.180	.003	3	5 *s*	4898.285	.014	2
4	4876.509	.008	3	3 *s*	4899.218	.008	2 *b N*
4	4876.871	.002	3	3 *s*	4899.45	.020	2 *N*
7 *s*	4877.331	.010	4 *b*	7	4899.991	.011	4
2	4877.92	.009	4 *b N*	4	4900.41	.018	2 *N*
4	4878.15	.023	4 *vb*	6 *s*	4901.074	.006	3 *N*
4 *s*	4878.869	.012	4	4 *s*	4901.366	.010	2
4 *s*	4879.124	.012	4	4 *s*	4901.61	.016	2 *b N*
5 *s*	4879.777	.012	1	11 *s*	4902.465	.007	6 *s*
3 *s*	4880.11	.042	1	7	4903.10	.019	2 *N*
5 *s*	4880.452	.014	2	5	4903.57	.022	4 *N*
5 *s*	4881.060	.012	4 *b*	8 *s*	4904.304	.005	4 *s*
3 *s*	4881.45	.021	3	5	4904.767	.011	1 *N*
3	4882.00	.012	2	7	4905.275	.013	4
2	4882.23	.004	2	9 *s*	4905.702	.009	4
3 *s*	4882.84	.029	3 *N*	5 *s*	4906.40	.024	0
3 *s*	4883.12	.027	3 *N*	7	4906.782	.013	0
3 *s*	4883.47	.007	1	7 *s*	4907.383	.011	6
4	4883.94	.016	2 *N*	6	4907.830	.012	1
3 *s*	4884.46	.042	1	5	4908.315	.009	3 *s*
4 *s*	4884.874	.005	4 *b*	5 *s*	4908.913	.009	2 *b d ?*
4 *s*	4885.112	.018	4	3	4909.796	.004	4
4	4885.737	.008	1	3	4910.155	.001	4
4	4886.03	.018	1	2	4910.49	.057	0
4	4887.08	.014	2 *b d*	4	4911.14	.015	1
4	4887.543	.008	3 *s*	5 *s*	4911.715	.013	4
4	4888.05	.027	1	4	4912.15	.022	4

TABLE I — *Continued.*

n	Wave-length	Mean error	Intensity and character	*n*	Wave-length	Mean error	Intensity and character
5 *s*	4912.744	0.008	1	4 *s*	4942.375	0.009	4
4 *s*	4913.060	.007	1	5 *s*	4942.905	.016	4
7 *s*	4913.433	.009	4	7 *s*	4943.703	.009	3 *N*
4	4914.089	.011	1	6 *s*	4944.338	.009	3 *b d ?*
4	4914.50	.019	1	18 *s*	4945.031	.004	6 *s*
6 *s*	4915.058	.012	3	4	4945.531	.018	4
5 *s*	4915.394	.015	2	3	4946.39	.027	4 *b*
6 *s*	4915.977	.009	2 *N*	7 *s*	4947.287	.006	4
6 *s*	4916.408	.010	2	7 *s*	4947.804	.007	5
7 *s*	4916.868	.013	2 *N*	4 *s*	4948.439	.003	1
4	4917.33	.017	2 *N*	3 *s*	4949.01	.026	1
6 *s*	4917.943	.011	7	2	4949.48	.002	1 *N*
4	4918.52	.018	1 *N*	3 *s*	4949.97	.018	2 *N*
3	4919.07	.007	1 *N*	2	4950.44	.006	1 *N*
3	4919.92	.024	3	4 *s*	4951.06	.016	4 *N* triplet?
3	4920.33	.006	2	4 *s*	4951.67	.026	2 *N*
3	4920.88	.029	2 *N*	2	4952.21	.002	2 *N*
5	4921.448	.006	4	2	4952.65	.018	2 *N*
4	4921.984	.015	4	4	4953.32	.017	6 *b*
2	4922.52	.011	2	3	4953.77	.018	0
4 *s*	4923.012	.010	2	6 *s*	4954.31	.018	4 broadened on red side
5 *s*	4923.530	.011	4				
2	4924.28	.010	4	4	4954.93	.017	4
3	4924.86	.019	3	2	4955.54	.005	1
2	4925.74	.030	2 *N*	4 *s*	4956.02	.018	3 *N*
3 *s*	4926.49	.004	2 *bN*	5 *s*	4956.477	.013	2
8 *s*	4927.213	.008	3	2	4956.79	.011	1
2 *s*	4927.69	.009	0	3	4957.10	.017	2
6	4928.149	.009	5	2	4957.62	.018	2
7 *s*	4928.796	.008	4 *b*	5 *s*	4958.204	.007	3
9 *s*	4929.547	.009	5 *b*	18 *s*	4958.736	.004	4 *s*
5	4930.397	.018	5 *b*	4 *s*	4959.366	.011	3
6 *s*	4931.144	.007	5	4 *s*	4960.13	.023	3 *b N*
2	4931.47	.015	0	5 *s*	4960.767	.008	4
2	4932.11	.016	1	3 *s*	4961.33	.013	1 *N*
6 *s*	4932.85	.019	4	3	4961.987	.003	2 *b*
3	4933.86	.013	1 *b d N*	2	4962.74	.019	3
5 *s*	4934.77	.012	2 *b N*	5 *s*	4963.277	.009	3
4 *s*	4935.123	.014	2 *N*	4 *s*	4964.233	.009	4 *b d*
4 *s*	4935.583	.010	2 *N*	3 *s*	4964.872	.014	2
4	4935.85	.017	3	4 *s*	4965.535	.012	4 *b d*
5 *s*	4936.339	.009	2	2	4966.11	.030	0
6 *s*	4936.695	.015	5	3 *s*	4966.597	.012	4
4	4937.20	.010	2	5 *s*	4966.977	.011	4
4 *s*	4937.727	.017	2	2	4967.81	.020	1
3	4938.61	.018	2 *N* haze	2	4968.24	.044	4 *N*
3	4939.03	.016	2 haze	3	4968.99	.009	2
3	4939.60	.011	2 *N*	7 *s*	4969.533	.009	6 *b*
4	4940.11	.020	2 *N*	2	4970.06	.023	2
7 *s*	4940.655	.007	4	3 *s*	4970.522	.013	6 *b*
4 *s*	4941.290	.010	2	2	4970.89	.037	2 *N*
6 *s*	4941.830	.013	4 *b*	2	4971.69	.002	2

TABLE I — *Continued.*

n	Wave-length	Mean error	Intensity and character	*n*	Wave-length	Mean error	Intensity and character
5 *s*	4972.252	0.007	4	2	5004.11	0.021	2
2	4972.77	.004	2	2	5004.95	.012	2 *N*
4 *s*	4973.264	.011	5	4 *s*	5005.34	.024	4 *N*
4	4974.294	.009	4 *N*	2	5006.25	.007	2
5	4974.912	.007	4 *s*	4 *s*	5006.92	.025	2 *N*
3	4975.48	.029	2 *b N*	4 *s*	5007.685	.010	3 *b N*
2	4976.54	.007	1 *N*	5 *s*	5008.594	.015	5 *v b*
5 *s*	4976.943	.014	2 *N*	2	5009.32	.018	1 *b N*
2	4977.43	.092	0 *N*	2	5009.81	.021	2 *N*
2	4978.211	.003	6	2	5010.154	.004	4
2	4978.76	.042	2 *b N*	2	5010.64	.100	0 *N*
2	4979.76	.019	3 *N d ?*	2	5011.20	.011	1
2	4980.42	.017	7	4 *s*	5011.637	.010	1
5 *s*	4981.114	.009	4	4	5012.104	.005	8 *s*
3	4981.67	.021	8	2	5012.78	.024	2 *N*
4 *s*	4982.337	.004	4	3	5013.59	.005	1 *b N*
2	4982.70	.001	2	2	5014.06	.025	1
3	4983.14	.015	2	3	5014.47	.016	4 *b*
2	4984.31	.019	5 *N*	2	5014.70	.050	0
2	4984.75	.017	0	3	5015.13	.021	2
2	4985.16	.009	3	5 *s*	5015.740	.007	4
3	4985.71	.008	3	2	5016.55	.031	5 *N*
3	4986.23	.020	5	2	5016.96	.030	3
5 *s*	4987.18	.015	2 *b*	2	5017.42	.029	0
5 *s*	4987.854	.008	3	3	5017.69	.024	5
3 *s*	4988.165	.012	2	2	5018.60	.019	3
5 *s*	4988.576	.011	6 *s*	2	5018.95	.013	3
2	4989.06	.022	1	5 *s*	5019.425	.009	6
2	4989.55	.003	3	2	5020.49	.038	4 *vb d ?*
3 *s*	4989.906	.010	3	5 *s*	5021.313	.016	4
4 *s*	4990.408	.011	4	2	5021.67	.015	1
4	4991.371	.010	7 *b*	2	5022.05	.025	2 *N*
4 *s*	4991.993	.015	4 *N*	2	5022.30	.014	2 *N*
6 *s*	4992.757	.012	6	2	5022.79	.014	4 *N*
3	4993.62	.033	5	2	5023.064	.002	4 *N*
2	4994.17	.017	2	3 *s*	5023.499	.012	5
4 *s*	4994.809	.004	5 *N*	2	5024.12	.017	1
6 *s*	4995.465	.007	3	5 *s*	5024.592	.008	5
4 *s*	4995.914	.023	5 *N*	2	5024.95	.002	5
3 *s*	4996.193	.003	5	2	5025.54	.017	1
3 *s*	4996.544	.010	2	2	5025.816	.002	3
2	4997.28	.007	1 *N*	2	5026.29	.002	1
5 *s*	4997.824	.010	4 *vb*	2	5026.66	.011	2 *b N*
7 *s*	4998.805	.010	8	3	5027.19	.010	2 *N*
4	4999.471	.017	2	3	5027.52	.016	2 *N*
4 *s*	4999.993	.017	2	3	5028.426	.005	2 *N*
4	5000.917	.013	2	2	5028.775	002	2 *N*
3	5001.38	.015	2	3 *s*	5029.235	.013	2
2	5002.04	.013	2 *b*	2	5029.87	.004	6
2	5002.75	.011	0	2	5030.22	.010	1
5 *s*	5003.359	.014	4 *b*	3 *s*	5030.657	.011	3
2 *s*	5003.61	.021	3	2	5031.11	.060	0 haze

TABLE I—*Continued.*

n	Wave-length	Mean error	Intensity and character	*n*	Wave-length	Mean error	Intensity and character
2	5031.46	0.008	2 *N*	3	5059.47	0.017	1
2	5031.90	.012	1 *N*	4	5059.800	.014	2
2	5032.42	.013	3	3	5060.56	.006	2 *N*
3 *s*	5032.87	.020	3 *N*	6 *s*	5061.383	.011	5 *b*
2	5033.566	.001	2	2	5062.20	.002	1 *b N*
3 *s*	5033.89	.009	2 *N*	2	5062.89	.004	1 *b N*
4 *s*	5034.541	.009	3	2	5064.08	.027	1 *N*
2	5034.89	.012	3	2	5064.49	.024	3
3	5035.50	.031	3	3	5065.39	.025	3
3	5036.07	.014	3	4 *s*	5066.110	.010	4 *s*
3	5036.77	.023	4	3	5066.65	.017	2 *N*
2	5037.19	.009	0	3	5067.23	.034	1 *N*
3	5037.56	.021	1	2	5067.86	.014	1 *N*
5 *s*	5038.028	.014	5	2	5069.11	.053	2 *vb N*
2	5038.57	.010	2 *N*	2	5069.60	.021	2 *N*
2	5039.35	.030	2	2	5070.59	.003	2 *N*
3 *s*	5039.824	.017	5	2	5071.45	.020	3
2	5040.17	.034	2	2	5071.90	.033	0 *N*
3 *s*	5040.746	.009	4 *s*	3	5072.46	.016	3 *b N*
2	5041.28	.014	2	2	5073.29	.014	1
2	5041.56	.030	2	2	5073.61	.030	0
2	5042.46	.001	1 *vb d? N*	3	5073.99	.007	2
2	5043.48	.034	1 *b N*	2	5074.65	.015	2 *b N*
3	5044.38	.018	2	2	5075.38	.008	3
4 *s*	5045.061	.010	3	2	5076.08	.023	3
4 *s*	5045.500	.012	3	2	5076.54	.015	3
2	5046.02	.014	1 *N*	2	5077.06	.015	2
2	5046.62	.036	0	6 *s*	5077.434	.009	5
3 *s*	5047.22	.014	3	2	5078.01	.009	1
3 *s*	5047.78	.012	2	4 *s*	5078.543	.012	4
3	5048.24	.012	3	2	5079.08	.018	1
2	5048.94	.008	4	2	5079.57	.006	6 *s*
3	5049.33	.019	2	2	5080.39	.005	2
2	5049.68	.002	2 *N*	2	5080.78	.015	2
2	5050.05	.022	2 *N*	2	5081.69	.007	1 *b N*
2	5050.71	.003	3	2	5082.28	.008	1 *b N*
2	5051.01	.011	3	2	5082.83	.009	2
2	5051.48	.016	2	2	5083.50	.008	1 *b*
3	5051.87	.010	4	2	5084.15	.010	4
2	5052.33	.012	3	2	5084.78	.012	2
2	5052.65	.006	3	2	5085.62	.040	0
2	5053.15	.009	2 *N*	2	5086.25	.001	3
2	5053.650	.000	2 *N*	2	5086.95	.027	1
3 *s*	5054.21	.023	3	1	5087.22		2
2	5054.60	.015	2 *N*	2	5087.93	.035	2 *b N*
2	5055.18	.032	3	2	5088.72	.018	3 *vb d*
2	5055.59	.014	2 *N*	2	5089.53	.004	2
2	5056.36	.013	3	4 *s*	5090.416	.006	4
2	5056.93	.035	3	2	5091.10	.007	1
6 *s*	5057.519	.012	4 *b*	2	5091.58	.001	0
3	5058.08	.026	1	3	5092.17	.024	2 *N*
6 *s*	5058.647	.009	4 *b*	2	5092.86	.033	2 *N*

TABLE I — *Continued.*

n	Wave-length	Mean error	Intensity and character	*n*	Wave-length	Mean error	Intensity and character
2	5093.39	0.010	1	2	5122.13	0.025	1
4 *s*	5093.962	.004	3	2	5122.72	.010	1 *N*
2	5094.755	.003	3 *vb*	2	5123.38	.072	1 *b N*
2	5095.86	.028	4	2	5124.57	.002	2 *N*
2	5096.22	.025	1 haze	2	5125.11	.008	2 *N*
2	5096.68	.005	1 haze	2	5125.93	.014	1
3	5097.22	.017	2	2	5126.38	.038	0 *N*
3	5097.66	.022	2 *N*	3	5126.89	.015	3
3	5098.22	.009	2	3	5127.51	.012	2
2	5098.75	.005	2 *N*	2	5128.08	.027	3 *s*
2	5099.05	.025	2 *N*	2	5128.45	.003	1
2	5099.53	.008	1 *N*	3	5128.94	.008	3 *s*
2	5099.86	.011	1 *N*	3	5129.59	.013	2
3	5100.27	.026	2	3	5130.35	.022	2 *b N*
2	5100.77	.057	1 *N*	4	5131.327	.012	8
2	5101.12	.017	3	2	5131.95	.001	2 *N*
5 *s*	5101.85	.017	3	4	5132.240	.014	2 *N*
2	5102.55	.035	1 *N*	2	5132.864	.001	1
3	5102.86	.018	2 *N*	5 *s*	5133.372	.013	4
2	5103.31	.040	0	3	5134.52	.021	2 *vb N*
2	5104.06	.023	2	4 *s*	5135.765	.015	5
3 *s*	5105.15	.024	4 *b*	5	5136.95	.023	3 *N*
2	5105.915	.002	3	2	5137.51	.025	0
3	5106.21	.015	4	5	5138.173	.012	3 *vb*
2	5107.26	.009	4 *b d?*	5	5139.15	.019	4 *b*
2	5108.33	.038	5	4	5139.81	.017	3
2	5108.77	.006	0	3	5140.61	.026	3 *b N*
2	5109.15	.008	0	4	5141.08	.017	2 *N*
2	5109.69	.002	4	3	5141.71	.012	2
2	5110.07	.013	4	2	5142.45	.026	2 *N*
2	5110.61	.005	2	2	5142.75	.033	2 *N*
2	5111.13	.023	1 *N*	5	5143.283	.014	2
2	5111.54	.022	2 *N*	2	5143.69	.027	1 *N*
2	5112.11	.004	2	5	5144.372	.013	4 *vb*
2	5112.66	.030	1	2	5145.32	.004	0
2	5113.08	.016	2	6	5146.027	.010	4 *b*
2	5113.49	.014	2	6	5146.741	.015	3
2	5113.87	.002	1	6	5147.597	.009	4 *b*
2	5114.24	.006	1	2	5148.69	.015	0
2	5114.66	.005	1	7	5149.531	.010	2 *N*
2	5114.97	.004	1	3	5150.56	.010	1
2	5115.81	.018	1 *N*	2	5151.12	.006	4
3	5116.22	.004	1 *N*	2	5151.34	.020	4
3	5116.60	.005	1 *N*	5	5151.81	.019	0
2	5117.40	.036	4 *b*	6	5152.421	.008	4
2	5117.85	.008	4 *b*	6	5153.042	.015	3
2	5118.69	.024	0 *N*	4	5153.712	.012	2
2	5119.24	.011	4	5	5154.22	.018	2 *N*
2	5119.78	.007	4	4	5154.955	.013	4 *N*
2	5120.45	.011	0	4	5155.38	.021	2 *b N*
2	5121.07	.009	4	2	5156.33	.022	2 *b*
2	5121.46	.067	0 *N*	5	5157.059	.012	4 *N*

TABLE I — *Continued.*

n	Wave-length	Mean error	Intensity and character	*n*	Wave-length	Mean error	Intensity and character
5	5157.553	0.013	3 *N*	4	5161.67	0.026	2
3	5158.19	.027	1	5	5162.37	.021	1 *N*
4	5158.52	.019	4	6	5162.99	.016	7
6	5159.113	.011	4	3	5163.40	.011	0
2	5159.75	.006	0	5	5163.948	.010	3
3	5160.53	.013	0	5	5164.954	.015	3
3	5161.20	.022	1 *N*				
Chlorine at atmospheric pressure				Chlorine at a pressure of $2\frac{1}{2}$ atmospheres			
3	5165.584	.015	2	3	5165.596	.012	3
3	5165.987	.008	2	4	5166.002	.016	3
4	5166.396	.005	2	2	5166.428	.015	3
5	5167.255	.007	5 *vb*	5	5167.268	.010	10 *vb*
11	5168.426	.006	6 *s*	8	5168.427	.005	9 *s*
3	5168.821	.013	1 *N*				
3	5169.733	.008	5 *b*	6	5169.727	.008	10 *b*
3	5170.998	.015	2 *vb N*	4	5171.008	.022	5 *b*
4	5172.014	.013	3	5	5172.039	.007	3
3	5172.419	.005	3	4	5172.461	.013	3
3	5172.91	.017	0				
4	5173.437	.013	2	6	5173.433	.008	2
4	5173.946	.010	2	6	5173.947	.013	2
4	5174.741	.016	1 *vb N*	4	5174.756	.018	2 *b*
4	5175.499	.008	4 *s*	5	5175.515	.015	5 *s*
3	5176.142	.008	4	6	5176.149	.008	5
2	5177.109	.003	3	4	5177.142	.014	4
3	5177.615	.009	2 *N*	3	5177.618	.015	4
3	5178.30	.029	0				
4	5178.780	.015	6 broadened on red side	6	5178.818	.005	7 *b*
3	5179.197	.031	0 haze	3	5179.375	.027	1 band
3	5179.692	.029	0 haze				
4	5180.550	.019	3 *b N* broadened on red side	4	5180.612	.010	5 *b*
1	5181.101		0 *N*	3	5181.074	.022	1 *N*
3	5181.619	.007	0	3	5181.631	.009	1
3	5182.379	.018	3 *b N* broadened on red side	6	5182.413	.012	7
3	5182.927	.010	0 *N*	1	5182.914		1 *N*
2	5183.408	.003	0 *N*	1	5183.366		1 *N*
3	5184.244	.022	3 *b N* broadened on red side	6	5184.300	.013	8
3	5184.555	.014	0				
4	5184.999	.024	1	4	5185.014	.020	1
4	5185.579	.006	1				
4	5186.142	.011	8	6	5186.181	.006	10
3	5186.517	.007	0	1	5186.528		0
3	5187.271	.017	2 *b*	3	5187.328	.020	2 *b*
3	5188.158	.021	3 *b*	3	5188.122	.012	6 *b*
4	5189.028	.019	1 *b*	6	5189.024	.016	2

TABLE I—*Continued.*

Chlorine at atmospheric pressure				Chlorine at a pressure of 2½ atmospheres			
n	Wave-length	Mean error	Intensity and character	*n*	Wave-length	Mean error	Intensity and character
3	5189.862	0.026	1 *N*	3	5189.894	0.013	1 *N*
3	5190.314	.004	1 *N*	4	5190.285	.025	1 *N*
2	5190.816	.013	0 *N*	4	5190.826	.030	0 *N*
3	5191.395	.005	1	4	5191.495	.016	2
3	5191.964	.007	1	5	5191.928	.015	2
3	5192.533	.003	3	6	5192.536	.011	4
3	5193.540	.011	2 *b*	5	5193.512	.010	3 *b*
2	5194.158	.013	2	4	5194.154	.012	2
4	5194.706	.013	5	5	5194.751	.014	8
2	5195.444	.006	2	4	5195.452	.014	2
3	5196.220	.018	5 *s*	6	5196.219	.011	6 *s*
2	5196.916	.031	3 *N*	6	5196.913	.017	4 *N*
2	5197.320	.010	1 *N*	2	5197.340	.005	1 *N*
2	5197.839	.008	1	3	5197.906	.011	2
2	5198.314	.003	1 *N*	2	5198.300	.015	1 *N*
3	5198.832	.027	1 *s*	6	5198.841	.011	7 *s*
				3	5199.465	.019	3
3	5199.836	.009	3 *N*	4	5199.854	.011	6
3	5200.625	.018	2 *N*	1	5200.648		4
3	5201.076	.014	2 *N*	1	5201.012		4
3	5202.108	.017	3 *vb*	3	5202.073	.018	6 *vb*
3	5203.150	.010	4 *b*	3	5203.166	.015	8 *b*
2	5204.414	.020	4 *b*	4	5204.407	.010	10 *b*
3	5205.722	.018	2 *vb*	4	5205.729	.026	5 *b*
3	5207.113	.043	2 *vb*	4	5207.100	.005	5 *vb*
3	5207.915	.013	1	4	5207.942	.011	2
				3	5208.479	.040	0
3	5209.423	.020	2	3	5209.440	.029	3
2	5210.188	.014	1 *N*	3	5210.186	.010	2 *N*
				4	5210.754	.018	2 *b N*
2	5211.677	.043	1 *N*	4	5211.703	.016	2
3	5212.537	.017	0 *N*	3	5212.355	.019	2
1	5213.13		1 *N*	4	5213.348	.014	3 *bd N*
2	5213.491	.006	1 *N*				
2	5214.998	.058	1 *N*	2	5214.051	.046	2 *b N*
2	5215.896	.028	1 *N*	4	5215.892	.019	2
2	5216.886	.050	2 *b N*	4	5216.917	.009	5 *b*
1	5217.899		1	4	5217.874	.012	2
2	5218.783	.009	2 *b*	4	5218.755	.017	5 *b*

Chlorine at 2½ Atmospheres Pressure.

6	5219.715	.016	1	5	5227.316	.013	2
5	5220.696	.010	10	5	5227.73	.016	1 *N*
6	5221.72	.015	2	5	5228.63	.025	3
4	5222.73	.019	5 *b*	5	5229.105	.010	2
4	5223.87	.008	1 *b N*	5	5230.051	.010	3 *b N*
4	5224.62	.035	3 *vb N*	5	5230.61	.025	1 *N*
4	5225.40	.017	1	5	5231.16	.025	1 *N*
5	5226.37	.021	3 *vb*				

TABLE I — *Continued*

Chlorine at $2\frac{1}{2}$ atmospheres pressure.

n	Wave-length	Mean error	Intensity and character	*n*	Wave-length	Mean error	Intensity and character
3	5231.55	0.028	0	4	5269.44	0.020	1
7	5232.12	.02	2	4	5270.09	.021	1 *N*
5	5233.37	.010	6 broadened	4	5270.808	.009	2 *N*
			on red side	5	5271.408	.012	2 *N*
2	5233.74	.025	2	2	5272.20	.008	0
5	5234.093	.006	7 *s*	4	5273.04	.023	3 *N*
5	5236.324	.011	4 *b*	4	5273.60	.023	0
5	5237.309	.006	5	4	5274.27	.026	0
4	5238.10	.018	4 *b d* ?	3	5274.67	.040	0 *N*
7	5239.05	.014	1	5	5275.27	.020	2 *N*
4	5240.15	.024	2 *N*	5	5276.05	.022	1
5	5240.55	.014	2 *N*	5	5276.653	.015	4
3	5241.41	.018	0	6	5277.342	.014	4 *s*
6	5241.997	.019	3 *b*	7	5278.040	.008	4
6	5242.518	.011	2	7	5278.962	.013	1
5	5242.930	.014	3	4	5279.80	.015	2 *b*
5	5244.151	.029	1 *b N*	4	5280.38	.019	2 *N*
7	5245.110	.010	3 *s*	5	5280.981	.006	2 *N*
6	5245.973	.012	2 *s*	4	5281.32	.024	1 *N*
5	5246.47	.028	1 *N*	4	5282.091	.011	3
5	5247.13	.022	1 *N*	5	5282.549	.012	2
6	5247.936	.010	3	4	5283.249	.012	2
5	5248.635	.009	5 *b*	4	5283.70	.015	3
5	5249.527	.014	1	4	5284.39	.020	2 *N*
6	5250.061	.013	4	4	5284.86	.018	2 *N*
5	5250.72	.019	0	3	5285.63	.010	0
5	5251.13	.032	1	3	5286.22	.014	1
4	5251.72	.016	2	4	5286.94	.056	0 *N*
5	5252.26	.015	1	4	5287.53	.023	2 *N*
5	5253.282	.008	3 broadened	4	5288.25	.023	1
			on red side	5	5288.808	.012	3
5	5254.606	.016	3 *b*	5	5289.62	.019	2
5	5256.26	.023	1	5	5290.342	.007	5 *s*
5	5257.08	.019	3	5	5291.131	.008	1
3	5257.41	.005	1	6	5291.863	.008	4
6	5258.166	.015	4 *s*	5	5292.701	.009	1
5	5258.882	.013	4 *b*	4	5293.522	.010	4
6	5260.088	.009	4	4	5294.48	.020	1 *b*
7	5260.584	.009	4	5	5295.268	.012	4
3	5261.79	.024	1 *N*	5	5296.11	.022	0
5	5262.068	.010	3	6	5296.97	.016	4
4	5262.649	.012	2	5	5297.92	.019	1 *b*
5	5263.376	.007	3	4	5298.75	.017	1 *b*
5	5263.82	.023	2 *N*	5	5299.60	.014	2
4	5264.77	.040	1 *vb N*	4	5300.65	.020	4 *b N*
4	5265.73	.037	0	3	5301.50	.015	1
5	5266.310	.010	1	3	5302.48	.018	1 *b N*
4	5266.96	.023	2 *N*	3	5303.02	.064	1 *b N*
4	5267.46	.016	2 *N*	3	5304.69	.018	1 band
4	5268.517	.006	2 *N*	3	5306.90	.024	1 *N*
4	5268.98	.022	2 *N*	3	5307.49	.055	1 *vb N*

TABLE I — *Continued.*

Chlorine at 2½ atmospheres pressure

n	Wave-length	Mean error	Intensity and character	n	Wave-length	Mean error	Intensity and character
2	5307.90	0.046	0 *N*	4	5346.70	0.021	2
3	5308.43	.027	0 *N*	4	5347.40	.016	1
4	5308.99	.017	1	4	5347.960	.006	1
4	5309.62	.017	0	4	5348.702	.013	2
3	5310.11	.018	2	4	5349.556	.005	1
5	5310.958	.013	2 *s*	4	5350.48	.043	0 *N*
4	5311.718	.013	3 *s*	4	5351.63	.021	1 *N*
4	5312.57	.033	0 *N*	4	5352.650	.011	1 *b*
4	5313.42	.025	1 *N*	4	5353.878	.014	1 *b*
4	5314.13	.016	2	4	5355.04	.021	1 *b*
4	5315.074	.005	1	4	5355.98	.022	1
4	5315.63	.026	1	5	5356.76	.016	1
2	5316.36	.043	0	3	5357.29	.017	1
3	5316.66	.011	1	4	5357.52	.022	1
2	5317.28	.005	0	3	5358.25	.020	1 *N d*
4	5317.968	.013	1	4	5359.02	.018	1 *N*
3	5318.44	.029	0	5	5359.476	.007	1
4	5319.17	.014	1	4	5360.38	.016	2 *vb*
3	5319.78	.016	1	5	5361.88	.022	1 *vb N*
3	5320.55	.025	3	5	5363.58	.032	1 *b N*
3	5320.89	.020	3	5	5365.03	.010	1
4	5322.19	.017	3 *b d?*	4	5366.05	.009	0
2	5322.76	.017	0	5	5366.71	.021	1
4	5323.608	.010	4 *b*	3	5367.43	.015	0
4	5324.42	.014	0	2	5367.98	.003	0
4	5325.18	.018	1	4	5368.575	.006	1
4	5326.721	.010	3	2	5369.667	.012	0
4	5327.49	.023	1 *b*	4	5370.18	.022	0
4	5328.438	.014	4	4	5370.64	.053	1 *N*
4	5330.07	.016	5 *b*	5	5371.21	.025	0 *N*
3	5330.65	.017	0	4	5372.03	.018	2
4	5331.909	.009	4 *s*	5	5372.61	.012	2 *b*
3	5332.58	.021	0	3	5373.44	.014	1
4	5333.24	.019	0	4	5374.30	.015	2 *b*
6	5333.638	.011	5 *s*	5	5375.28	.017	1
4	5334.65	.031	0 *N*	5	5376.25	.019	1
4	5335.224	.014	1	5	5377.10	.021	1
4	5335.74	.023	1 *N*	3	5378.29	.063	1 *vb N*
4	5336.59	.015	0	5	5380.49	.022	1 *b N*
4	5337.80	.025	2 *b N*	3	5381.98	.033	0 *N*
3	5338.60	.034	0	4	5382.92	.014	0
4	5339.21	.042	0	3	5383.52	.020	1
4	5339.90	.017	1 *N*	3	5384.55	.022	2 *b*
4	5340.82	.049	0 *N*	2	5385.74	.011	0
4	5341.49	.056	0 *N*	2	5386.83	.030	0
4	5342.27	.018	2 *b d N*	3	5387.48	.016	1 *b*
5	5343.18	.015	1	2	5387.93	.028	1 *N*
2	5343.68	.040	0 *N*	2	5388.71	.008	0 *N*
2	5344.42	.063	0 *N*	3	5389.19	.051	0 *N*
4	5344.88	.022	0 *N*	2	5389.85	.008	0
4	5346.02	.021	2	3	5390.36	.028	1

TABLE I — *Continued.*

n	Wave-length	Mean error	Intensity and character	*n*	Wave-length	Mean error	Intensity and character
2	5391.22	0.044	0 *N*	1	5423.15		0 *b N*
3	5392.13	.022	1 *b N*	1	5425.12		0
2	5393.88	.063	0 *vb N*	1	5425.87		0 *N*
3	5395.39	.042	1 *b N*	1	5426.71		0 *N*
3	5396.74	.011	1	1	5427.34		1
2	5398.03	.100	1 *b N*	1	5428.10		1
3	5399.50	.012	0 *b*	1	5429.79		1
2	5401.09	.032	0	1	5432.27		0
2	5402.37	.026	0 *N*	1	5433.95		0
2	5403.62	.023	0 *b N*	1	5434.44		1
2	5405.09	.011	0 *b N*	2	5435.46	.013	1 *b*
2	5405.69	.001	1	2	5436.98	.004	1 *b*
2	5406.73	.020	1	2	5438.21	.018	1 *b*
2	5407.36	.015	1 *N*	2	5438.86	.015	0
2	5408.39	.057	0	2	5439.56	.011	0 *b*
2	5409.02	.006	1	2	5440.77	.005	0 *b*
2	5409.74	.023	0	2	5442.09	.064	0
2	5410.22	.026	0	2	5442.77	.005	1
2	5410.85	.035	0	2	5443.46	.010	0
2	5412.04	.042	1 *N*	2	5444.14	.007	0
2	5412.75	.047	1 *N*	1	5444.67		0
2	5414.65	.053	0 *N*	1	5445.40		0
2	5416.41	.100	0 *b* band	2	5448.42	.005	0
1	5418.37		0 band	2	5449.26	.018	0
1	5420.31		0 band	2	5450.05	.002	0

The line spectrum.—The lines of the absorption spectrum of one meter of chlorine at atmospheric pressure begin at about $\lambda 4799$ and extend to $\lambda 5350$; the first lines in the blue are very faint, as are also, in general, those in the yellow above $\lambda 5200$. Those of maximum intensity lie nearer to the blue end than to the yellow. Few of the lines are sharp, the greater number have ill-defined edges, and not many are very broad. There are numerous pairs of lines, and some groups of three lines; but there is no apparent law in the distribution of these groupings.

The spectrum, especially in the first order, presents to the eye a fluted appearance, or recurring maxima and minima of absorption, which becomes more distinct as the quantity of chlorine is increased. The flutings are narrow at the point where the lines begin in the blue and become gradually broader on going towards the red. Examined under the microscope

they do not appear to have heads like those of the carbon bands, nor have they any visibly regular grouping; but the lines in the part of the band which gives the impression of greatest absorption to the eye are somewhat closer together than those in the rest of the band. The lines have all about the same width and intensity. It is possible that this fluted appearance may be due in great part to narrow patches of general absorption superposed on the line spectrum.

The general absorption spectrum.— The general absorption spectrum of chlorine consists of a broad band in the violet which was early discovered and which has been studied by Liveing and Dewar. There are also, perhaps, some smaller patches of absorption superposed on the region containing the line spectrum and giving to it a fluted appearance, as described above. The width of the violet band varies with the pressure and length of the column of chlorine used, and some measurements of it will be given later.

The effect of pressure on the general absorption spectrum.— It is generally considered that the absorption band spectrum of a gas arises from the line of spectrum as the result of increasing the density of the gas in question. Hasselberg,[1] describing the spectrum of nitrogen peroxide, says, that with increase of density new lines appear, not previously visible, and that the old ones become stronger, until together they form broad bands; finally, through increase in the number, blackness and width of the lines, the absorption becomes total. In the case of iodine, he says that with increase of density or of length of column, all the lines increase in intensity and width, and that at the same time a continuous absorption is developed, beginning at the violet end of each fluting and extending towards the red; and that finally the absorption becomes total. Konen[2] describes the same effect on the spectrum of iodine. In emission spectra it is generally considered that a line spectrum may be changed into a continuous spectrum by sufficiently increasing the pressure. It

[1] *Mém. de l'Acad. de St. Petersb.* [7], **26**, No. 4, 17, 1878; **36**, No. 17, 8, 1889.

[2] *Wied. Ann.*, **65**, 285, 1898.

might, therefore, be thought possible to change from a continuous absorption spectrum to a line spectrum by decreasing the pressure. This might take place in two ways: either the continuous absorption would break up into a number of lines scattered over, approximately, the same region that the general absorption occupied, thus being the reverse of what happens in the visible spectrum of iodine and nitrogen peroxide when the density is increased; or, the edges of the general absorption might recede until only one narrow absorption band or line would be left.

Some experiments were made to find what effect the decrease of pressure would have on the general absorption band in the spectrum of chlorine.

The tube 1.37 m long being exhausted, chlorine was let in until the total pressure, as measured by a manometer attached, was 30 cm of mercury in one case, and 10 cm and 5 cm in other cases. Further, the steel tube 33.2 cm long was exhausted and filled with chlorine at 10 cm pressure. Photographs were made of the absorption spectrum in each case. In the ultraviolet part the tube 80 cm long fitted with quartz ends was used with the gas at pressures of an atmosphere, half an atmosphere, and about 8 cm. In no case was any trace of a breaking up into lines found. The experiments were not entirely conclusive in this respect in the region of the ultra-violet below the absorption band, for the arc spectrum in this part is not sufficiently continuous to make the decision final as to the nonexistence of absorption lines, and the difficulties of photographing became too great beyond $\lambda 2500$, owing to the increase of the diffuse light reflected from the grating. With the chlorine at atmospheric pressure the general absorption extends to $\lambda 4700$; at 5 cm it has receded to $\lambda 3800$. In the region $\lambda 4700$–$\lambda 3800$ careful search could, therefore, be made for lines; but none were found. The total effect of a decrease of the pressure of chlorine was a diminishing of the width of the absorption band; hence we may consider the general absorption as a broadened line. If we could sufficiently decrease the pressure we might

expect to find this line. It broadens much more rapidly on the less refrangible than on the more refrangible side with increase of pressure, and finally extends into the region of the absorption lines. The same effect on the width of the absorption band may be obtained by decreasing the length of the column of chlorine, and leaving the pressure unchanged as by decreasing the pressure with a given length of column.

The edges of the band are not sharply defined. So that determinations of its width can have only relative accuracy. The following are some measurements for different pressures of chlorine, which are reduced to the equivalent columns at atmospheric pressure.

TABLE II.

Equivalent length of column at atmospheric pressure	Width of absorption band
272 cm.	λ4990 to ....
103	λ4700 " ....
60	λ4650 " λ2599
40	λ4420 " λ2630
14	λ4115 " λ2750
7	λ3850 " λ2881
3.5	λ3718 " ....

The effect of pressure on the line spectrum.—Photographs were taken of the visible line spectrum with the 1.37 m tube filled at 30 cm and 10 cm pressure. In the first case the lines in the blue and green-blue appeared unchanged in position and perhaps a little sharper than when chlorine at atmospheric pressure was used, but the lines in the green-yellow were too faint to be measured. In the second case, with 10 cm pressure, all the lines had become faint.

In order to study the effect of increasing the pressure, a new steel tube 1.36 m long was made, fitted with caps the openings in which were 4.9 cm in diameter. The end glass plates were very carefully ground to the ends of the tube; but notwithstanding the care taken, great difficulty was experienced in using the tube. It held sufficiently well, however, for pressures up to 2½ atmospheres, that is, when the partial pressure of the

chlorine was about 2 atmospheres, and it was used at this pressure. The spectrum so obtained showed no new lines in the part where they had been measurable before, but the old ones were much more intense. The broadening of the lines was slight and not nearly so noticeable as their marked increase in intensity. Farther up in the yellow new lines became visible; at the same time the general absorption advanced up to $\lambda 4990$, so that those lines which had been previously the strongest and sharpest were no longer visible. This spreading out of the general absorption did not appear to come from broadening of the lines, for as far as lines could be seen, which was right down to the edge of the general absorption, they remained hardly less sharp and distinct than before. This absorption, then, is rather to be considered the extension of the general absorption band from the violet up into the green, and the lines become invisible because of the greater intensity of the general absorption.

It seems as if we must consider that there are two absorption spectra here, each of which is separate and distinct from the other. The one is a line absorption, which persists always, though in certain cases it may be invisible, because it is too faint, or, on the other hand, because it is covered over by the general absorption. The other is a general absorption which consists of a broad band in the violet end of the spectrum, and possibly also of little patches, which lend, partly, the appearance of channelings to the line spectrum. With increase of density the lines become stronger, and are unchanged in position; but the general absorption broadens out very much, especially towards the less refrangible end, until it engulfs the lines and renders them no longer visible. The presence of two spectra might be explained by the different action which different groupings of the constituents of the gas may have on light radiations. One could think of the general absorption as due to molecule-complexes, and of the line absorption as due to simpler combinations of the separate atoms. An analogy in emission spectra is the spectrum of the arc light, in which brilliant lines are seen on a continuous bright background, like the

superposition of a line spectrum on a continuous one. In the case with which we are dealing it is not to be expected that we could see the absorption lines on the black background with the light used.

As the question of the fixed position of the lines under changes of pressure is of interest, measurements are given in Table I on the lines from $\lambda 5165$ to $\lambda 5218$ for chlorine at atmospheric pressure and at two and a half atmospheres pressure. A comparison shows that, except for some lines broadened on the red side, the displacement is, in general, smaller than the error of measurement, and is sometimes toward the blue end and sometimes toward the red. The conclusion is that the apparent differences are due to errors of measurement and differences in setting. It is generally accepted that within moderate limits an increase of temperature has the same effect on the absorption spectra of gases as an increase in pressure. The present experiments have all been made at room temperatures.

COMPARISON OF CHLORINE SPECTRA.

Morren's description of the general appearance of the chlorine absorption spectrum agrees with the results of the present experiments. The region over which lines have been observed has been extended, although the quantity of chlorine has been smaller than that used by Morren. This may be explained by the fact that his observations were made directly with the eye. The truth of the remarks of Gernez concerning the chlorine spectrum is not borne out by the results now found, since, according to the present investigation, 4.68 meters of chlorine (the amount used by Gernez) would absorb a large part of the visible spectrum, and would not give a spectrum extending into the "violet," as his statement reads. There are no measurements given in either case with which to compare the present ones.

The general absorption of 15 mm of liquid chlorine, as given by Gänge, corresponds roughly to that of a meter of chlorine gas at two and a half atmospheres pressure, as the numbers given above for the beginning of the total absorption show.

The results of the experiments here made on the total absorption band in the violet agree with those obtained by Liveing and Dewar. Strict comparison of the measurements on the width of the band is impossible, as Liveing and Dewar do not give details concerning the amounts of chlorine used.

It was not to be expected that the chlorine absorption spectrum would agree with its known emission spectrum. The phenomenon of reversed lines shows that bodies which emit light of certain wave-lengths will also absorb light of those same wave-lengths; and through certain ranges it has been shown that the absorption or emission is independent of the temperature, provided the general mode of vibration remains unaltered. If, however, the mode of vibration changes, owing, perhaps, to some change in the molecular or atomic grouping, the emission at one temperature is not the same as the absorption or emission at another. In the case of iodine, comparison has been made by Konen[1] between the absorption spectrum and the various emission spectra; he states that the absorption spectrum corresponds to the band spectrum of vacuum tubes, to the flame spectrum, and to the glow spectrum obtained by heating the gas in closed vessels.

In the present case the only known chlorine emission spectrum is a line spectrum. The lines lie between $\lambda 3276$ and $\lambda 6758$, but the majority are at the violet end, which region is covered in the absorption spectrum of a meter of chlorine by a broad band. In the region which contains lines in both emission and absorption spectra, coincidences were looked for. A large number of lines are nearly coincident, as will be seen from the following table, which contains the wave-lengths and intensities of the emission lines as given by Eder and Valenta,[2] together with the wave-lengths and intensities of the lines lying nearest to them in the absorption spectrum. Except for a few cases the differences are too great to be due to errors in measurement and the relative intensities are altogether different.

[1] *Loc. cit.*, p. 259. [2] *Loc cit.*, p. 8.

TABLE III.

λ_e	I_e	λ_a	I_a	λ_e	I_e	λ_a	I_a
4810.194	9	4809.949	1	5158.9	½	5159.113	4
4819.628	9	4820.059	2	5162.50	1	5162.37	1
4896.905	5	4896.888	2	5173.4	1	5173.43	2
4904.905	4	4904.767	1	5176.0	½	5176.14	4
4917.870	2	4917.943	7	5189.74	1	5189.86	1
4924.90	1	4924.86	3	5193.6	½	5193.54	2
4927.3	½	4927.213	3	5218.07	3	5217.87	2
4943.1	½	4942.905	4	5221.48	4	5221.72	2
4970.3	1	4970.52	6	5285.8	½	5285.63	0
4995.7	1	4995.914	5	5392.300	4	5392.13	1
5078.361	4	5078.543	4	5423.441	6	5423.15	0
5083.59	1	5083.50	1	5423.703	2		
5089.6	½	5089.53	2	5443.587	5	5443.46	0
5099.36	1	5099.53	1	5444.412	3	5444.14	0
5103.18	4	5102.86	2	5445.12	1	5445.40	0
5113.3	1	5113.08	2				

A comparison of the absorption spectra of iodine and bromine with that of chlorine shows that with decreasing atomic weight, increasing amounts of the gas must be used to render visible and distinct the absorption lines. Hasselberg obtained the absorption spectrum of iodine using a column of 10 cm of the vapor, and that of bromine by using a 75 cm column, whereas 137 cm of chlorine was not more than sufficient for the purpose. The given iodine spectrum extends a little farther on the red side than the bromine spectrum, and much farther than that of chlorine even for a 136 cm column at two atmospheres pressure; that is, the lines of this group shift towards the red with increasing atomic weight. This is analogous to the behavior of the emission spectra of any one group of elements, as shown by Kayser and Runge. Hasselberg measured about 3000 lines in the iodine spectrum, and about 2500 in the bromine spectrum; in the spectrum of chlorine only about 1000 have been observed. This number would probably have been increased if still greater amounts of chlorine had been used.

The lines of the iodine spectrum are, according to Hasselberg, fully as sharp and distinct as the solar lines; those of bromine, it is to be inferred, are not so good; and certainly the

chlorine lines are much less sharp and well defined than those of the solar spectrum.

As previously mentioned, the earlier investigators observed a similarity in the appearance of the iodine and bromine spectra, but under the dispersion now used, the lines which they observed are seen to be bands composed of a number of lines. These bands appear as channelings in the spectra and continue to make the two spectra appear somewhat similar. These, as already noted, are not absent from the chlorine spectrum, but again iodine has them most distinctly, and bromine and chlorine in diminishing distinctness. If one used the proper densities in all three cases the three spectra could doubtless be made to assume a very similar appearance.

SUMMARY OF RESULTS.

The complete absorption spectrum of chlorine at ordinary temperatures consists of a very broad total absorption band in the violet region, of a line absorption in the blue, green and yellow, and of transparency in the visible red.

The lines do not coincide with the known line emission spectrum of chlorine.

With increase of pressure the absorption band in the violet broadens out rapidly on the less refrangible side, more slowly on the more refrangible side. Decrease of pressure does not break it up into lines.

The total absorption band behaves as if it were a separate absorption spectrum from the line spectrum.

Increase of pressure increases the intensity of the line spectrum greatly and causes new lines towards the red to become visible.

The absorption lines of the elements of the halogen group shift toward the red with increasing atomic weight.

The number and sharpness of the absorption lines of the elements of the halogen group increase with increasing atomic weight.

The amount of gas necessary to render visible the absorption

lines of the elements of the halogen group decreases with increasing atomic weight.

In conclusion, it is a pleasure to me to express my deep gratitude to Professor A. Stanley MacKenzie under whose direction this investigation has been carried out, for the constant encouragement and assistance which he has given me during the course of this work.

PHYSICAL LABORATORY,
BRYN MAWR COLLEGE,
June 1901.

ON THE PERIOD OF A ROD VIBRATING IN A LIQUID.

By Mary I. Northway and A. Stanley Mackenzie.

As early as 1786 Buat announced in his "Principes d'Hydraulique" that the period of a pendulum was greater when vibrating in a fluid than when vibrating in a vacuum, not only because of the loss in weight due to buoyancy but also because of the mass of fluid which must be considered as participating in the motion of the pendulum. The latter is loaded by, or drags with it, a certain amount of the fluid. Buat determined this added mass for the case of a sphere and for bodies of other forms. Little attention was paid to this work until Bessel[1] about forty years later, in determining the length of the seconds pendulum, concluded from theoretical reasons that it was necessary to take into account the inertia of the air as well as its weight. When the dimensions of the bob were small in comparison with the length of the suspending wire, Bessel represented the apparent increase in the weight of the pendulum by a constant, k, times the mass of the displaced fluid. For a sphere 2 in. in diameter vibrating in water he found for k the value 0.9459; later he changed this to 0.956.

Sabine[2] tried to find the effect of air on a pendulum by making it vibrate in air and then in a vacuum. He found that the old correction for buoyancy should be multiplied by a factor m, whose value was 1.655, in order to obtain the total correction.

In 1832 Baily[3] published the results of some similar experiments. Four spheres of about 1.5 in. in diameter gave for m the value 1.84; and three spheres of about 2 in. in diameter gave 1.748. For spheres of about the same size Bessel had found the value 1.956. For small cylinders Baily found the value of m to increase regularly with decrease of diameter, but according to no apparent law.

[1] Abh. d. Akad. d. Wiss., Berlin, 1826; Coll. de Mem. Soc. Fr. de Phys., **5**, 1.

[2] Phil. Trans., 1829; Coll. de Mem. Soc. Fr. de Phys., **5**, 134.

[3] Phil. Trans., 1832, p. 399; Coll. de Mem. Soc. Fr. de Phys., **5**, 185.

Poisson[1] determined analytically the value of m for spheres and found it to be 1.5; which agrees very well with the results of Buat's experiments made fifty years earlier.

In 1843 Stokes[2] found from theoretical considerations that for a long cylinder oscillating in a direction perpendicular to its length the effect of the inertia of the fluid was the same as if a mass equal to that of the displaced fluid were distributed along the axis of the cylinder. This agrees very well with the experimental results obtained by Baily for a long tube 1.5 in. in diameter.

Up to this time no law was known connecting the variation of m with the dimensions of the pendulum, and the only property of the fluid taken into consideration was its density. In 1848 Stokes[3] treated the problem of a body oscillating in an infinite liquid medium. He dealt with the cases of a disk rotating about its axis, of a sphere oscillating in the direction of a diameter, and of an infinitely long cylinder vibrating in a direction perpendicular to its length. He considered the effect of the viscosity as well as of the density of the fluid. This paper is of especial interest here because the experiments about to be described approximate more nearly to this problem than to any other that has been treated mathematically. The main results of this important piece of work will be referred to later.

In 1880 Montigny[4] published an account of some experiments which he had made in 1859 in order to find the effect of the density and of the compressibility of various liquids on the pitch of bells. (He intended these experiments to be preliminary only and to repeat them with electrically driven forks. He evidently did not do so, although in a second paper[5] he describes his method of maintaining forks in vibration in a liquid, and states that the pitch is lower the denser the liquid.) He found the pitch of the bell first in air, and then when submerged mouth upward in the liquid; also when in air and filled to the brim with the liquid; and finally when

[1] Mem. de l'Acad. de Paris, 11, 1831.

[2] Trans. Camb. Phil. Soc., **8**, 105, 1843; B. A. Report, 1848.

[3] Trans. Camb. Phil. Soc., **9**, pt. 2, p. 35; Coll. de Mem. Soc. Fr. de Phys., **5**, 277.

[4] Bull. de l'Acad. de Belg., [2], **50**, 159, 1880.

[5] Bull. de l'Acad. de Belg., [2], **50**, 300, 1880.

empty and immersed to the brim in the liquid. The pitch was the same for the last two cases, and the lowering of pitch was less than when the bell was entirely submerged; from this he concluded that the lowering was due to that part of the liquid which was in contact with the bell. The lowering was found to depend also on the shape and material of the bell. The pitch was determined by comparing the note by ear with that of a monochord. He found that the lowering of pitch was greater the denser the liquid and the graver the pitch. He gives the following intervals of lowering[1] for four of the bells used.

	Sol_4	Ut_5	Sol_5	Ut_6
When filled with water . .	1.204	1.125	1.127	1.118
When submerged in water .	1.411	1.327	1.286	1.261

Auerbach,[2] in a paper written in 1878, attempted to deduce a simple theoretical relation between the pitch of a tuning fork in air and that in a liquid as a consequence of the different way in which kinetic energy is dissipated in the two cases. The pitch, he contends, always depends on the square root of the coefficient of elasticity, and this can have two entirely different values according as the vibration takes place isentropically or isothermally. The former he considers to be the case for air and the latter for liquids. He then takes the coefficients of elasticity in the two cases to be as the ratio of the specific heats, 1.4 : 1, and the interval of lowering of pitch to be as $\sqrt{1.4} : 1$; that is, as 1.18 : 1. He tried four forks in air and in water, observing the pitch by ear, and obtained the following intervals:

Frequency of fork in air,	132	264	396	528
Interval of lowering,	1.11	1.12	1.13	1.15

He concluded from these experiments that the ideal interval, 1.18, would be approached more and more nearly the higher the pitch. He adds that experiments with other liquids show that the density and viscosity of the liquid have no influence on the lowering of pitch.

[1] The ratio of the pitch in air to that in the liquid is commonly called the interval of lowering.

[2] Wied. Ann., 3, 157, 1878.

Koláček,[1] referring to this work by Auerbach, suggests that an explanation of this lowering of pitch can be given from simple mechanical principles, and that the effect of the liquid is mainly to load the fork by a certain quantity which is the product of the density, ρ, of the liquid and a factor, c, depending on the size and shape of the vibrating body. Accordingly he finds the interval of lowering to be $\frac{T'}{T} = \sqrt{1 + c\rho}$. Taking the mean, 1.13, of the intervals found by Auerbach, he determined the value for c to be 0.277. Using this value of c it follows that Auerbach's forks should give $\frac{T'}{T}$ equal to 2.18 when made to vibrate in mercury, and 1.21 when in sulphuric acid. On plunging a 435 fork into various liquids and noting its changed pitch by an ear comparison with a monochord, Koláček found the value of $\frac{T'}{T}$ to be 2.1 for mercury, 1.21 for sulphuric acid, and about the same for ether and alcohol as for water. He proceeds to treat analytically the simple cases of the radial dilatational and linear translational oscillations of a sphere in a liquid, and finds expressions for $\frac{T'}{T}$ in the two cases. When applied to a solid iron sphere vibrating in water these expressions give for $\frac{T'}{T}$ the values 1.03 and 1.17 respectively. Neither of these cases is that of a tuning fork, but they give limiting values for it and the mean of the two agrees roughly with Auerbach's results. In a later paper Koláček[2] gives a more rigorous hydrodynamical discussion of the general problem of the vibration of a solid in a liquid.

In 1882 Auerbach[3] experimented with glass cylinders filled with liquid. Assuming that the effect of the liquid was to load the cylinder with a mass, c, depending on the nature of the liquid and on the shape and dimensions of the cylinder, he deduced the equation, $\frac{T'}{T} = \sqrt{\frac{m+c}{m}}$, for the lowering of pitch, where m is the mass of the cylinder. He obtained the following results: (1) the interval was

[1] Wied. Ann., 7, 23, 1879.

[2] Sitzb. math.-naturw. Cl. Wien, 87, Abth. 2, 1147, 1883.

[3] Wied. Ann., 17, 964, 1882.

smaller the higher the pitch; (2) the interval was independent of the height of the cylinder; (3) the interval was greater in proportion to the narrowness of the cylinder; (4) using different liquids whose densities varied from 0.729 to 1.364, and in addition mercury, he found that as a first approximation the specific lowering of pitch (the ratio of the lowering for any liquid to that for water) depended on the density only and increased with it, but more slowly, and as a second approximation that the specific lowering increased as the compressibility of the liquid decreased. He determined the pitch by Kolácek's method.

Miss L. R. Laird,[1] working in this laboratory, made some experiments on the change in the period of a pianoforte wire 0.446 mm. in diameter and 107 cm. long, vibrating in the following liquids: water, solutions of potassium carbonate, and mercury. The wire was made to vibrate by means of an electro-magnet and its record was taken on a revolving drum along with that of a standard tuning-fork. The results for the lowering of pitch agreed very well with the values obtained analytically by Stokes in his 1848 paper for the case of an infinite cylinder. The change of pitch gave the following intervals: 1.06 for water; 1.13 for potassium carbonate of density 1.47; 1.095 for potassium carbonate of density 1.22; 1.72 for mercury. In the case of water the observed values agreed still more closely with numbers calculated on the simple supposition that the wire was loaded with the mass of the displaced liquid, which was the conclusion arrived at by Stokes in his 1843 paper. Some experiments on a similar wire 0.933 mm. in diameter and 37 cm. long gave the interval 1.065 for water, 1.18 for oil of density about 0.9, and 1.12 for a sodium nitrate solution of density 1.38. It will be noticed that the lowering for water was the same for both wires.

The following experiments were made in continuation of those of Miss Laird, and were intended to separate, if possible, the effects of the density and the viscosity of the liquid. The change in pitch of a tuning fork is of most direct interest, and, as the nearest simple approach to this, a rod clamped at one end was studied. As has been stated, this problem is a very rough approximation to the case of the infinite cylinder discussed by Stokes. The present problem

[1] Phys. Rev., 7, 102, 1898.

differs from his in that the rod is of a different cross section and has edges, which greatly changes the stream lines; that it has not the same amplitude at all points, and is of finite length. Nevertheless, a consideration of Stokes' work will be of great service, and will throw at least some light upon the way in which the constants of the liquid must enter.

If we suppose that a mass m has a simple oscillatory motion as a whole in the x direction, and is in a vacuum, its equation of motion is

$$(1) \qquad m\frac{d^2x}{dt^2}+px=0;$$

and its period

$$(2) \qquad T=2\pi\sqrt{\frac{m}{p}}.$$

Stokes found that the effect of a liquid medium is to introduce two terms into the above equation of motion. The first of these is of the nature of an acceleration, the second of a velocity. The former is equivalent to an increase of the mass of the vibrating body by a certain fraction of the mass of the displaced fluid; m becomes $m+km'$ where m' is the mass of the displaced fluid. This is the same as saying with Buat that the body drags along with it a definite quantity of the liquid. The quantity k will evidently be determined by the form of the stream lines and must accordingly be a function of the shape and size of the cross-section of the vibrating body, of the density of the liquid, and of the coherence of its separate layers, or of its viscosity; and this Stokes found to be the case. The second of the terms which he found to enter into the equation of motion is a retarding force proportional to the velocity. The form of this term can be assigned from general considerations; for the energy absorbed by the friction will be determined by the amount of the fluid displaced, and will depend not only on the velocity but also on the number of times its direction is changed. Stokes found it to be of the form $k'm'n$, where n is the pitch of the body in vacuum, and k' a quantity depending on the dimensions of the body and on the density and viscosity of the liquid. For the infinitely long cylinder he found, when $\frac{2}{a}\sqrt{\frac{1}{\pi n}\cdot\frac{\mu}{\rho}}$ was small,

(3) $$k = 1 + \frac{2}{a}\sqrt{\frac{1}{\pi n} \cdot \frac{\mu}{\rho}}, \quad \text{and} \quad k' = \frac{2}{a}\sqrt{\frac{1}{\pi n} \cdot \frac{\mu}{\rho}},$$

and hence $k = 1 + k'$ where a is the radius of the cylinder.

The quantity $\sqrt{\frac{1}{\pi n} \cdot \frac{\mu}{\rho}}$ enters also into the values of k and k' found for the disk and the sphere.

Our equation of motion (1) can now be written

(4) $$(m + km')\frac{d^2x}{dt^2} + k'm'n\frac{dx}{dt} + px = 0.$$

The well-known solution of this equation gives for the period T' the value

$$T' = 2\pi\left[\frac{p}{m + km'} - \frac{1}{4}\left(\frac{k'm'n}{m + km'}\right)^2\right]^{-\frac{1}{2}}$$

$$= 2\pi\left(\frac{m + km'}{p}\right)^{\frac{1}{2}}\left[1 + \frac{1}{8}\frac{k'^2m'^2n^2}{p(m + km')}\right]$$

approximately, assuming k'^2 to be small. This can be put in the form

$$T' = 2\pi\left(\frac{m}{p}\right)^{\frac{1}{2}}\left[1 + k\frac{\rho}{\Delta}\right]^{\frac{1}{2}}\left\{1 + \frac{1}{8}k'^2\left(\frac{\rho}{\Delta}\right)^2\frac{1}{1 + k\frac{\rho}{\Delta}}\right\}$$

where Δ is the density of the body. Using equation (2) this becomes

$$\frac{T'}{T} = \left(1 + k\frac{\rho}{\Delta}\right)^{\frac{1}{2}}\left\{1 + \frac{1}{8}k'^2\left(\frac{\rho}{\Delta}\right)^2\frac{1}{1 + k\frac{\rho}{\Delta}}\right\}$$

If k' is at all small we may neglect the second factor, especially for liquids of small density; in this case the interval of lowering of the pitch becomes

(5) $$\frac{T'}{T} = \sqrt{1 + k\frac{\rho}{\Delta}}.$$

This is equivalent to neglecting in our equation of motion the term which depends on the velocity. The following experiments were

made to test this equation and to find the form and value of the quantity k. One could approximate more closely to Stokes' theoretical problem than was done in the present investigation by using rods of circular cross-section; but the mechanical difficulties of maintaining such rods in vibration in a plane are too great, and those used had a rectangular cross-section, and so corresponded to one-half of a tuning-fork. Although in the theoretical discussion T is the period in a vacuum, it is a sufficiently good approximation for this investigation to take it as the period in air.

The rod to be experimented on was held in a heavy clamp screwed to the inside of a wooden box, which held, when required, the liquid under consideration. The rod was maintained in vibration by an electro-magnet which could be suspended at any height from a movable bar across the top of the box. The circuit was completed through the rod and a piece of platinum soldered to its free end dipping into a mercury cup fitted with a plunger and projecting up from the bottom of the box. When the rod was of brass a thin piece of soft iron 9 mm. wide and 32 mm. long was soldered to it just beneath the electro-magnet. The vibrations were recorded on a revolving drum, upon which a marker also recorded seconds as given by a standard clock. The pitch of a given length of the rod used was found first with the vibrations taking place in air; then without disturbing anything the box was filled with the liquid under consideration, and the lowering of pitch calculated. The box was large in order that the nearness of its walls might not influence appreciably the pitch of the rod. Three boxes were used, as follows:

	With Water.	With Oil.	With Sodium Nitrate Solution.
Internal length.	57 cm.	55 cm.	55 cm.
" breadth	26 "	21 "	25 "
" depth	30 "	22 "	22 "

To show that the size of the box is of great importance it may be stated that the pitch of the rod A (described below) in water was increased nearly five per cent. when it was changed from the large box to one whose dimensions were $52 \times 4.5 \times 4.5$ cm. In order to

prevent as far as possible the absorption of energy by the box, etc., the clamp was made of a mass of metal weighing fifteen pounds; the whole apparatus was placed on a stone pier; and the box, the cross-bar supporting the electro-magnet, and the projecting end of the rod, if any, were heavily weighted down with masses of iron. Unless this were done the pitch was sensibly affected.

The duration of contact of the platinum wire with the mercury, the distance of the electro-magnet and piece of soft iron from the clamp, the shape of the electro-magnet and its height above the rod, and the strength of the driving current were all found to affect the pitch of the rod. The time of contact used was always the shortest that would maintain the rod in vibration. The distance of the electro-magnet from the clamp affected the pitch in the following way: the shorter the distance the greater was the interval of lowering, no matter what was the vibrating length of the rod. Table I. gives a specimen of the readings taken for the purpose of studying this question; they were made on the rod *A* with half of its length (21.1 cm.) vibrating.

Table I.

Distance of Electro-magnet from Clamp.	Frequency in Air.	Frequency in Water.	Height of Electro-magnet above Soft Iron.	Interval of Lowering.	Average Interval.
2.1 cm.	44.67	38.45	3 mm.	1.162	
" "	44.63	38.40	3 "	1.162	1.162
" "	44.62	38.37	3 "	1.163	
11.0 "	38.89	33.63	8 "	1.153	
" "	39.09	33.72	8 "	1.156	1.154
" "	39.09	33.91	8 "	1.153	
17.5 "	36.92	32.21	8 "	1.146	1.147
" "	36.95	32.22	8 "	1.147	
19.5 "	36.64	32.09	8 "	1.142	1.142
" "	36.63	32.08	8 "	1.142	

The final average intervals, found as above, for various lengths of the rod were:

Average distance of electro-magnet,	6 cm.	11 cm.	19 cm.
Average interval,	1.160	1.153	1.148

The interval, therefore, diminishes as the driving force is placed nearer to the free end of the rod. In order to get results undisturbed by this possible source of variation, the electro-magnet was

placed at a constant distance from the clamp throughout the whole series of experiments, the piece of soft iron being taken off the rod and soldered on in a new place whenever the vibrating length of the rod was changed. The distance from the clamp chosen was such that with the shortest length of rod used the soft iron was at the tip end. For the rod *A* this length was 10.55 cm. and for the other rods it was 21.1 cm. A set of observations was taken to find also the effect of the height of the electro-magnet above the rod. The electro-magnet first used was of the *U* shape. Its cores did not project beyond the bobbins, whose ends were each about 2 sq. in. in area. The effect of the height was not very noticeable for air, but for water a change of height from 7 to 23 mm. made the rod vibrate more slowly by more than 1 per cent. There was no doubt that with this electro-magnet near the rod, the coils acted partly as a wall, and they were replaced by a single coil whose core projected 15 mm. and was 6 mm. in diameter. With this electro-magnet it was found that a diminution of height increased the pitch, no matter how far the electro-magnet was from the clamp, but in no case by so much as 1 per cent. and usually by much less. It was evident, therefore, that the height must be kept constant throughout the experiments; but as the height of the electro-magnet was increased, the current also had to be increased, and it seemed desirable to keep the current also constant. The current strength chosen was the smallest that would drive the shortest length of rod used, and the height chosen was the greatest possible for that current and length.

Five rods were used during the investigation. Four of them, called *A*, *B*, *C* and *D*, were of brass, and one, *E*, of steel. *B* had approximately twice the thickness of *A* and the same width; *C* twice the width and the same thickness; and *D* twice the width and twice the thickness. *E* had approximately the same dimensions as *A*. The dimensions and densities of the rods were:

	A	*B*	*C*	*D*	*E*
Width. . . .	0.94 cm.	0.97 cm.	1.91 cm.	1.95 cm.	0.97 cm.
Thickness. .	0.32 "	0.65 "	0.34 "	0.67 "	0.33 "
Density. . .	8.56	8.56	8.70	8.56	7.79

The liquids used were water, cotton-seed oil, and a solution of sodium nitrate saturated at 15° C. The first two had approximately the same density but very different viscosities, while the first and last had approximately the same viscosity but quite different densities. The viscosities of the oil and the solution of sodium nitrate were found by the method used by Gartenmeister.[1] The temperature throughout was kept as nearly as possible at 17°.6 C. At this temperature the liquids have the following constants :

	Water.	Cotton-Seed Oil.	Sodium Nitrate Solution.
Density, ρ	1.000	0.921	1.355
Viscosity, μ	0.011	0.781	0.030

The tables which follow give the results obtained. All of the separate readings are not given, but the average frequency in air and in the liquid for each length, and the corresponding interval of lowering. In no case were less than two distinct sets of observations made on each length of the rod used. Ordinarily 1,000 vibrations for each observation were counted on the drum and the corresponding number of clock seconds, and the pitch calculated therefrom.

Table II.

Rod A. Dimensions, 42.2 × 0.94 × 0.32 *cm. In water.*

Vibrating Length.	Frequency in Air.	Number of Readings.	Frequency in Water.	Number of Readings.	Interval of Lowering.
Full length.	10.33	4	9.04	3	1.148
¾ "	18.16	2	15.78	2	1.151
⅝ "	25.58	2	22.27	2	1.149
½ "	39.10	2	33.95	2	1.152
⅜ "	66.73	2	58.09	3	1.149
¼ "	136.44	4	119.45	6	1.143
Average Lowering.					1.149

From this table it is seen that for a range of about 3½ octaves the lowering is independent of the pitch within the limits of accuracy

[1] Zeitschr. f. physikal. Chem., 6, 524, 1890.

attained. The same deduction can be made from the numbers in each of the other tables. The lowering here is a little less than a minor whole tone plus a small semitone $\left(\frac{10}{9} \times \frac{25}{24} = 1.157\right)$.

Table III.

Rod E. Dimensions, 42.2 $\times$ 0.97 $\times$ 0.33 *cm. In water.*

Vibrating Length.	Frequency in Air.	Number of Readings.	Frequency in Water.	Number of Readings.	Interval of Lowering.
¾ length.	24.86	2	21.48	2	1.157
⅝ "	35.62	2	30.71	2	1.159
½ "	53.54	2	46.20	2	1.158
Average Interval.					1.158

The lowering here again is independent of the pitch. The interval is exactly a minor whole tone plus a small semitone, and differs by less than 1 per cent. from the interval for the brass rod of approximately the same dimensions. It seems safe to conclude that no great importance is to be attached to the materials used for the rods, at least for such materials as are ordinarily used for tuning forks, bells and kindred musical instruments.

Table IV.

Rod B. Dimensions, 42.0 $\times$ 0.97 $\times$ 0.65 *cm. In water.*

Vibrating Length.	Frequency in Air.	Number of Readings.	Frequency in Water.	Number of Readings.	Interval of Lowering.
Full length.	19.80	2	18.29	2	1.083
¾ "	34.56	2	31.65	2	1.094
⅝ "	47.50	2	43.60	2	1.089
½ "	72.14	3	66.51	3	1.085
Average Interval.					1.088

The lowering is nearly a semitone plus a comma, or a great limma $(\frac{16}{15} \times \frac{81}{80} = 1.080)$, and is not far from being one-half of that for rod *A* of one-half the thickness.

TABLE V.

Rod C. Dimensions, 42.2 × 1.91 × 0.34 *cm. In water.*

Vibrating Length.	Frequency in air.	Number of Readings.	Frequency in Water.	Number of Readings.	Interval of Lowering.
Full length.	10.12	2	7.95	2	1.273
¾ "	17.81	2	14.01	2	1.274
⅝ "	25.05	2	19.78	2	1.267
½ "	37.86	5	30.32	5	1.244
Average Interval.					1.265

The lowering is exactly a major third plus a comma ($\frac{5}{4} \times \frac{81}{80} = 1.265$) and is nearly twice as much as that for rod *A* of one-half the width.

TABLE VI.

Rod D. Dimensions, 42.0 × 1.95 × 0.67 *cm. In water.*

Vibrating Length.	Frequency in Air.	Number of Readings.	Frequency in Water.	Number of Readings.	Interval of Lowering.
Full length.	19.89	3	17.19	3	1.157
¾ "	35.13	8	30.29	7	1.160
⅝ "	48.23	2	42.34	2	1.140
Average Interval.					1.152

The lowering here is a little less than a minor whole tone plus a small semitone, and is almost exactly the same as that for rod *A* of one-half the width and one-half the thickness, as we should expect from the results shown in Tables IV. and V.

GENERAL APPROXIMATE RESULTS.

1. The interval of lowering for a rod of given cross section is independent of the length.

2. The interval of lowering for a rod of given cross section is approximately the same for brass and steel and is probably independent of the material within the range of substances ordinarily used.

3. The interval of lowering for a rod of given width is approximately inversely proportional to the thickness.

4. The interval of lowering for a rod of given thickness is approximately directly proportional to the width. More definite relations will be given later.

The above experiments were then repeated with cotton-seed oil instead of water. This oil was used because it is inexpensive, and consequently can be used in large quantities, and has its coefficient of viscosity about seventy times that of water. With this liquid it was hoped that the importance of the viscosity in the lowering of pitch could be determined. Whereas in the case of water the contact breaker was under the liquid, it had to be outside for the oil and the nitrate solution. The platinum wire was replaced by a platinum-tipped thin steel wire which reached above the surface of the liquid where it was bent over and dipped into the mercury cup. No special difficulty was experienced in maintaining the rod in vibration in the viscous liquid. The results of these experiments were as follows:

TABLE VII.

In Cotton-seed Oil.

Rod.	Vibrating Length.	Frequency in Air.	Number of Readings.	Frequency in Oil.	Number of Readings.	Interval of Lowering.	Av. Interval of Lowering.
A	Full length.	10.00	10	8.67	10	1.140	
"	¾ "	17.32	2	14.81	2	1.169	
"	⅝ "	24.57	2	20.91	2	1.175	
"	½ "	37.15	4	31.90	4	1.164	1.162
B	Full length.	20.26	3	18.42	4	1.105	
"	¾ "	36.11	3	32.22	3	1.121	
"	⅝ "	49.67	4	44.72	3	1.111	
"	½ "	65.18	3	59.65	3	1.093	1.108
C	Full length.	10.21	4	8.13	4	1.256	
"	¾ "	18.38	3	14.33	3	1.283	
"	⅝ "	25.76	3	19.90	3	1.294	
"	½ "	38.60	2	30.58	2	1.262	1.272
D	Full length.	19.67	3	16.89	3	1.166	
"	¾ "	35.30	2	29.54	5	1.195	
"	⅝ "	48.43	3	41.44	3	1.169	1.176

We shall discuss these numbers more carefully later, but we can say at once in a general way that the effect of viscosity is not very marked. In order to throw still more light on the effect of viscosity as compared with that of density, the four brass rods were made to vibrate in a solution of sodium nitrate, which had a viscosity of only 0.0299, and so not very much greater than that of

water, and whose density was 1.355, about a third as much again as that of water. These determinations resulted as follows :

TABLE VIII.

In Sodium Nitrate Solution.

Rod.	Vibrating Length.	Frequency in Air.	Number of Readings.	Frequency in Nitrate Solution.	Number of Readings.	Interval of Lowering.	Av. Interval of Lowering.
A	Full length.	10.00	2	8.47	3	1.181	
"	¾ "	17.25	2	14.48	3	1.191	
"	⅝ "	24.70	2	20.46	3	1.207	
"	½ "	37.58	2	31.51	2	1.193	1.198
B	Full length.	20.35	2	18.10	2	1.123	
"	¾ "	35.75	2	31.84	2	1.123	
"	⅝ "	50.99	2	44.96	2	1.134	1.127
C	Full length.	10.21	2	7.61	2	1.343	
"	¾ "	18.51	2	13.76	2	1.347	
"	⅝ "	25.98	2	19.26	2	1.349	1.346
D	Full length.	19.97	2	16.57	2	1.206	
"	¾ "	35.30	3	29.51	3	1.196	
"	⅝ "	49.15	3	40.88	3	1.202	1.201

Comparing these numbers with those obtained for water it is seen that the lowering is increased very decidedly, and that the increase is not far from being directly proportional to the added density. It is evident therefore that the main factor in the lowering of pitch is the density of the medium, and that the effect of viscosity is relatively small.

Thus far the conclusions drawn from the investigation have been of an approximate and general nature; it remains to determine whether an expression for $\frac{T'}{T}$ can be found which will contain the results of every case investigated. Stokes' work can afford little direct help since he found the value of $\frac{T'}{T}$ to depend on n, as will be seen by reference to equation (3); whereas in the present investigation the interval of lowering is independent of the pitch within the limits of accuracy attained. Indirectly his work is of great service, and his results have been used already in the formation of the equation of motion (4). Since for the case he discussed, the value of k'

is considerably smaller than that of k, and since the results here obtained for oil show that the change of pitch due to friction is relatively small, it is allowable as a first approximation to neglect k', and consider the lowering of pitch to be expressed by equation (5),

$$\frac{T'}{T}=\sqrt{1+k\frac{\rho}{\Delta}}.$$

The following table contains the values of k found from this equation.

TABLE IX.

Values for k.

Medium.	Vibrating Length.	Rod A.	Rod B.	Rod C.	Rod D.	Rod E.
Water.	Full length.	2.697	1.478	5.396	2.895	
	¾ "	2.776	1.682	5.418	2.954	2.641
	⅝ "	2.737	1.589	5.263	2.561	2.678
	½ "	2.796	1.515	4.761		2.660
Average.		2.752	1.566	5.210	2.803	2.660
Oil.	Full length.	2.785	2.054	5.454	3.342	
	¾ "	3.407	2.385	6.101	3.978	
	⅝ "	3.537	2.178	6.369	3.407	
	½ "	3.298	1.809	5.596		
Average.		3.257	2.107	5.880	3.576	
$NaNO_3$	Full length.	2.483	1.642	5.152	2.858	
	¾ "	2.632	1.643	5.220	2.707	
	⅝ "	2.873	1.799	5.255	2.797	
	½ "	2.662				
Average.		2.663	1.698	5.209	2.787	

The quantity k can be a function of the constants of the liquid and the dimensions of the rod. Stokes has shown that μ and ρ enter always as a ratio, and further that in the equation for the lowering of pitch they enter as $\sqrt{\frac{\mu}{\rho}}$; an attempt was accordingly made to express k as a function of $\sqrt{\frac{\mu}{\rho}}$ and the cross section of the rod (the length has been shown not to enter). This it has been found can be done in several ways of about the same degree of satisfactoriness. The following four equations give values for $\frac{T'}{T}$

with various forms of the quantity k; w denotes the width of the rod, and t the thickness.

$$(a)\qquad \frac{T'}{T}=\sqrt{1+2.63\,\frac{w^{\frac{7}{8}}}{t^{\frac{3}{4}}}\left(1+\tfrac{1}{4}\sqrt{\frac{\mu}{\rho}}\right)\frac{\rho}{\Delta}}.$$

$$(b)\qquad \frac{T'}{T}=\sqrt{1+1.12\,\frac{w}{t^{\frac{3}{4}}}\left(1+\tfrac{1}{4}\sqrt{\frac{\mu}{\rho}}\right)\frac{\rho}{\Delta}}.$$

$$(c)\qquad \frac{T'}{T}=\sqrt{1+\frac{w}{t^{\frac{7}{8}}}\left(1+\tfrac{1}{4}\sqrt{\frac{\mu}{\rho}}\right)\frac{\rho}{\Delta}}.$$

$$(d)\qquad \frac{T'}{T}=\sqrt{1+0.91\,\frac{w}{t}\left(1+\tfrac{1}{4}\sqrt{\frac{\mu}{\rho}}\right)\frac{\rho}{\Delta}}.$$

The last of these equations is preferred on account of its simplicity, but its concordance with the observed values is not so good as is the case with the first, where the greatest deviation is 1 part in 110. The observed average values for $\frac{T'}{T}$ and those calculated from equation (d) are given in Table X., along with their differences.

TABLE X.

Rod.	Medium.	Water.	Oil.	$NaNO_3$.
	Obs.	1.149	1.162	1.198
A	Cal.	1.150	1.164	1.200
	Diff.	+ .001	+ .002	+ .002
	Obs.	1.088	1.108	1.127
B	Cal.	1.078	1.086	1.106
	Diff.	− .010	− .022	− .021
	Obs.	1.265	1.272	1.346
C	Cal.	1.271	1.295	1.358
	Diff.	+ .006	+ .023	+ .012
	Obs.	1.152	1.176	1.201
D	Cal.	1.148	1.163	1.198
	Diff.	− .004	− .013	− .003
	Obs.	1.158		
E	Cal.	1.162		
	Diff.	+ .004		

It will be seen that the agreement is as close as that of the separate values of $\frac{T'}{T}$ in any of the tables II. to VIII.

The above equations (*a*), (*b*), (*c*) and (*d*) were found from a close consideration of Stokes' work. It is worthy of remark, however, that an entirely empirical formula can be found which will give an equally good agreement with the results. The following table contains the observed values of $\frac{T'}{T}$, and those calculated from the equation

$$(e) \qquad \frac{T'}{T} = 1 + 0.45 \frac{w^{\frac{7}{8}}\rho}{t\,\Delta} + 0.03\,\mu.$$

Table XI.

Rod.	Medium.	Water.	Oil.	$NaNO_3$.
	Obs.	1.149	1.162	1.198
A	Cal.	1.157	1.168	1.212
	Diff.	+ .008	+ .006	+ .014
	Obs.	1.088	1.108	1.127
B	Cal.	1.079	1.096	1.107
	Diff.	− .009	− .012	− .020
	Obs.	1.265	1.272	1.346
C	Cal.	1.273	1.275	1.368
	Diff.	+ .008	+ .003	+ .022
	Obs.	1.152	1.176	1.201
D	Cal.	1.180	1.196	1.254
	Diff.	+ .028	+ .020	+ .053
	Obs.	1.158		
E	Cal.	1.155		
	Diff.	− .003		

In the work which has been done on this subject there is only one case where sufficient details are given to enable us to apply our equations to the values found for the lowering of pitch. Koláček[1] states that for a horseshoe magnet 340 mm. long, 12.5 mm. wide and 46 mm. thick vibrating in water the value of $\frac{T'}{T}$ was 1.03. Calculated from equations (*d*) and (*e*) the interval is 1.02.

[1] Loc. cit.

It was thought worth while to test equation (d) by applying it to the numbers found by Miss Laird, taking $\frac{w}{t}$ as equal to 1. It was found that the values thus determined for the interval of lowering in water agree with the observed values even better than do the values calculated on the simple supposition that the wire is loaded with the mass of the displaced fluid. For denser liquids, however, equation (d) does not agree so well as the equation of Stokes, since the lowering was found to depend on the pitch. The following table contains the frequencies in air and in liquid observed by Miss Laird, and the differences between the observed frequency in liquid and numbers calculated (1) from Stokes' equation, (2) from equation (d), and (3) on the assumption that the wire is loaded with the mass of the displaced liquid. These differences are called Δ_1, Δ_2 and Δ_3 respectively.

Table XII.

	Pitch in Air.	Pitch in Liquid.	Δ_1	Δ_2	Δ_3
Water.	73.8	70.1	− 2.9	− .4	− .6
	92.3	87.9	− 3.6	− .7	− 1.0
	105.5	99.4	− 2.8	+ .3	− .1
	113.4	107.7	− 3.8	− .5	− .9
	126.5	117.8	− 1.8	+ 1.7	+ 1.3
Solution of Pot. Carbonate.	73.8	65.1	− .8	+ 2.9	
	92.3	82.6	− 1.7	+ 2.4	
	105.5	93.9	− 1.2	+ 3.3	
	113.4	99.7	+ .1	+ 4.8	
	126.5	111.6	− .2	+ 5.2	
Mercury.	73.8	43.0	− 1.8	+ 2.8	
	92.3	53.3	− 1.3	+ 4.0	
	105.5	61.9	− 2.2	+ 3.5	
	113.4	65.8	− 1.5	+ 4.7	
	126.5	73.4	− 1.5	+ 5.2	

An attempt was made to see what would come from assuming that there was always attached symmetrically to the rod when in motion a constant mass of liquid of elliptical cross section. It will be seen from the way in which the quantity k was introduced that it is the ratio of the volume of this attached liquid to the volume of

the rod. Using Table IX. it was found that if the major axis of the ellipse were $2\,w$ and the minor axis $t + \frac{1}{2}\,w$, the periods calculated, on the assumption (used in this paper) that the lowering of pitch is due only to the added mass of liquid, agreed very well with the observed values.

Physical Laboratory, Bryn Mawr College,
May, 1901.

[Reprinted from the PHYSICAL REVIEW, Vol. XV., No. 1, July, 1902.]

ON MAGNETOSTRICTION IN BISMUTH.

By A. P. Wills.

IN his extensive paper on magnetostriction in iron and some other metals, Bidwell[1] describes a remarkable effect which he observed on magnetizing a bismuth rod. He employed a cast cylindrical rod 13.2 cm. in length and .7 cm. in diameter. It was magnetized lengthwise by a powerful solenoid. With a magnetizing force of 280 C. G. S. units an elongation was suspected; with 470 units it was quite perceptible. Upon increasing the field to 680 units the extension was roughly measurable and amounted to .3 ten millionths of the length of the rod; and with a field of 842 units the extension was 1.5 ten millionths of the length. From these results it appears that the effect was increasing much more rapidly than the field. The field was not increased beyond the last-mentioned figure.

Since bismuth is the only substance outside of the strictly magnetic metals in which this effect has been observed, it seemed desirable to study the behavior of bismuth in this connection under the action of much higher fields. Now in the case of bismuth an electromagnet can be advantageously employed. For, a short rod can be used as well as a long one, since the end effects can be neglected and the pole pieces can therefore be brought close enough together to produce the powerful field desired.

[1] Philosophical Transactions, Vol. 179, p. 205.

With this idea in mind I wrote to Hartmann and Braun of Frankfurt am Main for samples of their electrolytically prepared bismuth. They furnished these in the form of cast cylinders somewhat over 1 cm. in length and .5 cm. in diameter. I turned one of these until its length was 1.1 cm. and its diameter .52 cm. The main experiments were performed with this cylinder.

In the preliminary experiments, I had at my disposal a very powerful electromagnet belonging to the physics department of the Massachusetts Institute of Technology.[1] Later, at Bryn Mawr, I used a much smaller one.

The arrangement for measuring any possible change in length of the bismuth cylinder consisted of a system of two levers and a high power micrometer microscope.

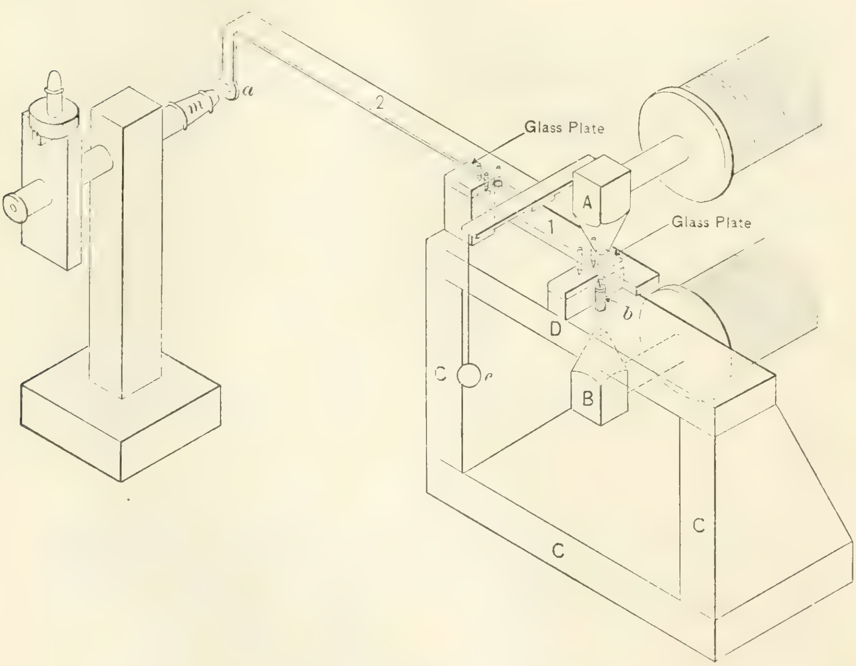


Fig. 1.

In Fig. 1 A and B are the pole pieces of the electromagnet; b is the bismuth cylinder; 1 and 2 are the two levers and m is the

[1] I wish to acknowledge here the courtesy of Professor Cross and Professor Wendell in putting the apparatus of the physical department at my disposal.

micrometer microscope; a is a portion of a microscope slide containing diatoms (upon one of which the microscope was focused): C is a heavy framework of wood and D is a heavy bar of brass. A circular recess was made at about the middle of this bar of such diameter that it made an easy sliding fit for the cylinder. The bottom of this recess was plane, so that the cylinder rested firmly upon it. A small bit of plane glass was shellacked to the upper face of the cylinder. Each of the levers played on three points. These points were made of sapphire and were highly polished. The levers themselves were of aluminium and quite light. One point on the first lever pressed against the glass on the end of the bismuth cylinder and the other two against a horizontal plate of glass firmly fixed to the supporting bar D. These three points were held properly in place by a small spring (not shown in the figure) pulling downward from a point midway between them. The tension on this spring could be varied at pleasure by turning a small windlass to which it was connected. At one end of the first lever there were two sapphire jewels firmly attached with shellac. One of these was a "V" jewel, the other a "slot" jewel. In these two of the points of the second lever played while the third rested on a glass plate firmly fixed to the bar D. There was a counterweight d and another (not shown in the figure) symmetrically placed which served to bring the center of gravity of the second lever in a vertical line about midway between the points. When in use, a box to protect the second lever from air draughts was found very desirable. All of the supports except that for the electromagnet were placed upon a brick pier. The support for the electromagnet was upon a table of its own, so that there was no material contact between the electromagnet and the apparatus on the pier. This was desirable, since the magnet supports would give slightly when the field was thrown on and any contact might cause a disturbance in the delicate lever system.

The lengths of the arms of the two levers and their multiplying powers were as follows:

	Short Arm.	Long Arm.	Multiplying Power.
Lever 1.	1.10 cm.	12.35 cm.	11.23.
Lever 2.	.41 cm.	24.0 cm.	48.4.

Hence the multiplying power of both together was $11.23 \times 58.5 = 657$. It was found that 1,162 head divisions on the micrometer eye-piece represented a length of one-tenth of a millimeter. Hence one head division would correspond to a change in length in centimeters of the bismuth cylinder equal to

$$\frac{1}{1162 \times 100 \times 657} = 1.31 \times 10^{-8}.$$

Now the maximum extension observed by Bildwell amounted to 1.8 ten-millionths of the length of the rod which was 13.2 cm. Hence the extension per centimeter of length would be 1.36×10^{-8} cm., and, in the present experiments, with a field of 1,000 an extension should have been detected.

In the experiments tried the field was varied, reaching a maximum of 3,200 C.G.S. units; but neither with the maximum field nor with any of the various intermediate fields could any effect be detected which was not due to one of the following causes: (1) A heat effect, in which case time was required to produce it. (2) An effect due to the action of the magnet on the currents induced in the levers themselves; in this case a sudden jerk was observed when the magnet circuit was made or broken and then the pointer would return to its zero position immediately. This effect was much more marked when the circuit was broken than when made.

These experiments was repeated with a bismuth cylinder made of bismuth which one buys in the market as chemically pure. But again negative results only were obtained.

After working with levers for some time it became evident that great care must be used if their action was to be relied on. The first system constructed was wrong somewhere as it was suspiciously non-susceptible to small vibrations and to small variations in temperature. The second lever was quite heavy and I suspected that the points were not right. So I made a new lever much lighter than the old and gave the points the arrangement shown in Fig. 1, except for the "V" and the "slot" jewels. Finally, however, I tried the point, slot, plane arrangement of the points and this im-

proved the action considerably, adding to the stability of the system without interfering with its sensitiveness. But in order to have a check upon the sensitiveness of the levers it was decided to measure the depression of a stiff steel spring under the action of small weights.

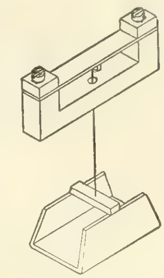


Fig. 2.

The spring was clamped firmly at both ends to a support (see Fig. 2) which was so designed that, when the bismuth cylinder was removed, it could be screwed to the brass bar *D* in such a way that the point of the first lever which rests on the cylinder could be made to rest on a bit of glass shellacked to the spring in the center. Below this point a string passed through holes in the supports and at the end of the string a stirrup for weights was attached.

The dimensions of the spring were as follsws :

Length,	2.70 cm.
Width,	.59 "
Depth,	.05 "

The theoretical depression of the center of the spring under known loads could be calculated from the formula for the case of a prismatic rod clamped at both ends with a load in the middle. In the calculations 23×10^{11} was assumed as the value of Young's modulus. This is roughly the mean of the values found in the tables. With this assumption a load of one gram should give a depression of 7.05×10^{-6} cm. But since there was a narrow strip of brass (for attaching the stirrup string) soldered transversely to the spring this value is a little larger than it really ought to be. The depression of the spring as actually measured was 6.70×10^{-6} cm. The agreement is quite satisfactory. Furthermore the depression was found to be proportional to the load. This seems good evidence that the lever system finally worked as it should. Further evidence is afforded by the fact that when the bismuth cylinder was in position and a lighted match held in its neighborhood the pointer in the microscope moved gradually and not jerkily.

The field was measured ballistically by means of a d'Arsonval galvanometer. The galvanometer was calibrated by means of a long standardizing solenoid.

It is intended to continue these experiments soon with a much more powerful electromagnet.

Physical Laboratory, Bryn Mawr College,
March 14, 1902.

ON SOME EQUATIONS PERTAINING TO THE PROPAGATION OF HEAT IN AN INFINITE MEDIUM.

BY A. STANLEY MACKENZIE.

(Plates XXIII–XXVIII.)

(*Read April 4, 1902.*)

We may attack a problem in the theory of the conduction of heat in two ways; we may make use of a Fourier's series or integral, or, since the general differential equation is a partial linear one, we may build up the required solution out of known solutions for simpler cases. The former way is usually much the more expeditious if the proper "trick" can be hit upon, but the method is a purely artificial one, throwing no light on the process involved. The student or reader sees at once that this method produces the required result and that a limited number of very similar problems might be treated in the same way, but he is apt to feel instinctively at first that the mathematical tool he has employed is one of which he has only a superficial knowledge and that will fail him when he gets out of a certain set of problems; he wonders what a Fourier's integral means and why it has a special value in such problems. The trouble here, as in many other departments of physics, is that the physical interpretation of mathematical operations is usually avoided. There can be but one good reason for this, since all must admit the desirability of such interpretations, that it is at times exceedingly difficult, if not impossible, to give the inherent physical meaning of a mathematical operation. Much more, however, might be done than is done, and there is perhaps no branch of mathematical physics more suited to the purpose of introducing to those just beginning such studies the meanings and the limitations of mathematical operations than heat conduction. The second method of treating heat conduction problems, by building up solutions from known solutions for other cases, is full of suggestiveness, and brings into view the meaning of many of the mathematical processes employed in any treatment of the conduction of heat, and the relationships of the equations involved. An attempt is made in the following pages to point out the necessity for effort along the lines indicated above, and among other things to give careful drawings of some of the more important curves of temperature and current.

Reprinted from Proceedings American Philosophical Society, Vol. XLI, No. 169.

In any heat conduction problem we have ordinarily three sets of equations, the general differential equation, the initial conditions, and the surface conditions. For the general purposes of this paper by taking the medium infinite we can get rid of the surface conditions without limiting the generality of the methods. Suppose we wish to study the case of a body of any shape or size maintained at any temperature in an infinite homogeneous medium of the same material as the body itself but initially at a uniform low temperature (which for convenience we take as the zero of temperature), or of the same body at a given initial temperature put into the medium and left to cool, we could find their solutions by an ordinary summation if we knew those for the corresponding problems in the case of an infinitesimally small particle. We might begin by assuming as Kelvin does (*Math. and Phys. Papers*, Vol. ii, p. 44), the solution for the case of a quantity of heat, Q, suddenly generated at a point $r = 0$ at time $t = 0$; but it will be better to see if it can be derived.

We have here to deal with the case of a symmetrical distribution of temperature about a point. The form of the general differential equation for this case is

$$\frac{1}{k}\frac{\partial V}{\partial t} = \frac{2}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where $k = \frac{K}{CD}$, K being the specific conductivity, C the specific heat, and D the density of the medium. This equation can be put in the more symmetrical form

$$\frac{1}{k}\frac{\partial (Vr)}{\partial t} = \frac{\partial^2 (Vr)}{\partial r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

This is of exactly the same form as that for the case of the "linear flow of heat" of Fourier, that is, of flow in one dimension only, namely,

$$\frac{1}{k}\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The distribution of Vr with reference to r for the case of symmetry about a point is the same as the distribution of V with reference to x for the case of symmetry about an infinite plane perpendicular to the axis of x. This fact will be of assistance in obtaining and translating results. The ordinary way of treating

any problem of spherical symmetry is to get the simplest kind of a solution of (1) or (2) and build up from that solution to the required one. There is of course an infinite number of solutions of these equations and a great many simple ones, but we can at once find one by trying $Vr = e^{ar + \beta t}$. This gives $\beta = ka^2$, and hence $Vr = e^{ar} e^{ka^2 t}$. Changing a to ia we get $Vr = e^{-ka^2 t}$ $(\cos ar + i \sin ar)$, and so a solution is

$$Vr = e^{-ka^2 t} \cos ar, \qquad (4)$$

where a is any constant. This equation represents a periodic distribution of Vr along a radius vector dying out with the time; for the case of the infinite plane this would be actually the curve of distribution of temperature along x. It is seen that the values of V in (4) possess maxima and minima; the temperatures are zero at distances given by $r = (2n + 1) \frac{\pi}{2a}$ at all times. There is a hot central sphere of radius $\frac{\pi}{2a}$, surrounded by alternate hot and cold shells of common thickness $\frac{\pi}{a}$, the maximum numerical temperature in each falling as we go away from the centre. Calling the thickness of the shells d, we have $a = \frac{\pi}{d}$; so that the constant a is inversely proportional to the thickness of the shells and determines it. The central point begins by being, and remains, infinitely hot; the hot and cold layers conduct heat to each other and gradually die down in temperature. At a great distance from the origin we should have practically the case of a medium made up of alternate hot and cold infinite plates of the same numerical temperature and the same thickness left to cool; and such a problem could be treated from a consideration of (4).

This case is far from the problem we started out to discuss. We can, however, get new solutions from the simple one above, and the common method is now to say that the following is a solution of (2),

$$Vr = \int_0^\infty e^{-ka^2 t} \cos ar \, da, \qquad (5)$$

and then translate this equation as we have just translated (4); but

instead of doing so we ought rather to be able to say that this operation means such and such and foretell the distribution of temperature it will give. This illustrates what was meant above when saying that we ought if possible to give the physical interpretations of mathematical processes. What is the meaning of the operation involved in (5)? Perhaps some light can be had on it from the following consideration: We are to take a series of distributions of temperature like that given by (4) and described above, where the constant a (determining the thickness of the shells) has the successive values, 0, da, $2da$, ... a, and superpose them on the medium after first reducing every temperature by multiplying it by da. We are then to take da indefinitely smaller and smaller, and finally to make a indefinitely greater and greater. We have thus the difficulty of a double limit entering, and if we wish to seek the initial condition it becomes a triple limit. This is sufficient to prevent any rash prediction in this problem as to the exact nature of the solution to be obtained; and this case serves as an excellent example of the difficulties to be overcome in any such efforts at physical interpretation. Before the limit is reached the state of temperatures is given by

$$Vr = da\left[1 + e^{-kt(da)^2}\cos rda + e^{-4kt(da)^2}\cos 2rda + \text{etc.}\right].$$

The limiting value of this series, which is equation (5), is not very evident without considerable study, but on account of the dying-out factor in each term the series is convergent, and the more rapidly convergent the greater the value of t, and its value could be found for any given t and da. Another way of finding this value at any time and distance required is to take an axis along which a's are measured and draw the logarithmic curve e^{-kta^2} and the curve $\cos ra$, then form the curve whose ordinate at each point is the product of the ordinates of these two curves at the point, and the area between this new curve and the axis gives the numerical value of Vr. Since this area is formed of pieces alternately above and below the axis of a and of decreasing numerical value, we see that Vr is always of the same sign and that, for any finite value of r, it begins by increasing in value and finally falls off to zero, and by inference that it is zero at time $t = 0$; but that at the origin it has initially a value greater than zero. The

operation (5) therefore promises at least another simple solution and one much nearer the desired one. Noting that

$$\int_{-\infty}^{+\infty} e^{-ka^2t} \cos ar\, da = 2\int_0^{\infty} e^{-ka^2t} \cos ar\, da, \text{ and that}$$

$$\int_{-\infty}^{+\infty} e^{-ka^2t} \sin ar\, da = 0, \text{ we get } \int_{-\infty}^{+\infty} e^{-(kta^2-ira)} da =$$

$$e^{-\frac{r^2}{4kt}} \int_{-\infty}^{+\infty} e^{-\left(\sqrt{kt}\,a - \frac{ir}{2\sqrt{kt}}\right)^2} da = \sqrt{\frac{\pi}{kt}}\, e^{-\frac{r^2}{4kt}},$$

and (5) becomes

$$Vr = \frac{A}{\sqrt{kt}}\, e^{-\frac{r^2}{4kt}}, \qquad (6)$$

where A is an arbitrary constant. This equation says that Vr is initially indeterminate (evidently infinite, from physical considerations) at the centre and zero elsewhere; as time goes on the value of Vr falls off indefinitely at the centre, rises to a maximum at all other points and then falls off indefinitely also. Now these are exactly the conditions we want for V itself for the case of an infinitely hot point cooling in an infinite medium initially of zero temperature. If we had been studying (3) we would have found the same equation as (6), with x for r and V for Vr, for an infinitely hot plane cooling in a medium initially zero. The form of the curves for Vr given by (6) is exhibited on Plates XXIII and XXIV; with values of r as abscissæ curves A^1 to A^4 are for values of the time $\frac{1}{16k}$, $\frac{1}{8k}$, $\frac{1}{4k}$ and $\frac{1}{2k}$ respectively; with values of $4kt$ as abscissæ curves B^1 to B^5 are for values of the distance 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$ and 1 respectively.

We have taken the form (2) of the differential equation in preference to (1) on account of its symmetry and because we are solving the case of the infinite plane at the same time; but it possesses another important advantage. Since either form of the equation is a linear partial one we can add any number of solutions for a new solution; the question arises, therefore, whether V being a solution $\frac{\partial V}{\partial r}$ and $\int Vdr$ are solutions, and what are their physical

meanings. Without thinking of the special form of the differential equation, we can find the meaning of $\frac{\partial V}{\partial r}$ as follows: Let a solution, V, be $f(r,t)$; then another, V_1, is $\frac{1}{\Delta r}f(r,t)$, where Δr is a small constant; and another, V_2, is $-\frac{1}{\Delta r}f(r,t)$. Superpose on the medium these two states of temperature, V_1 and V_2, after first displacing V_2 bodily to the positive side of the origin by an amount Δr. When Δr is indefinitely decreased the limiting state of temperature is that represented by $\frac{\partial V}{\partial r}$, or $\frac{\partial f(r,t)}{\partial r}$. That is, $\frac{\partial V}{\partial r}$ represents a heating due to a kind of doublet. We must next find out whether such a state of temperature as that represented by $\frac{\partial V}{\partial r}$ is a solution of (1). We saw that $\frac{\partial V}{\partial r}$ was a limiting case, and hence it is not a solution in the limit (except by some unusual accident) unless it is so just before the limit is reached. While Δr is still finite, but as small as we please, the superposed heatings do not satisfy the same differential equation; for V_1 satisfies the equation $\frac{1}{k}\frac{\partial f(r,t)}{\partial t} = \frac{2}{r}\frac{\partial f(r,t)}{\partial r} + \frac{\partial^2 f(r,t)}{\partial r^2}$, while V_2 satisfies the equation $\frac{1}{k}\frac{\partial f(r-\Delta r,t)}{\partial t} = \frac{2}{r-\Delta r}\frac{\partial f(r-\Delta r,t)}{\partial r} + \frac{\partial^2 f(r-\Delta r,t)}{\partial r^2}$, and on account of the variable coefficient these are not the same equation. Hence $\frac{\partial V}{\partial r}$ is not a solution of (1), and is only a solution of an equation in V when that equation has constant coefficients, that is, coefficients not containing r. Equation (2) is of that kind, and hence knowing a solution of it, Vr, we can say that $\frac{\partial (Vr)}{\partial r}$ is also a solution. Call this new solution $V_1 r$, then V_1 is a solution of (1). Since $\frac{\partial (Vr)}{\partial r} = V + r\frac{\partial V}{\partial r}$, and since $\frac{1}{r}\frac{\partial (Vr)}{\partial r}$ is a solution of (1), we have $\frac{V}{r} + \frac{\partial V}{\partial r}$ a solution of (1); this is what we have just called V_1. Now V satisfies (1), but we have just seen that $\frac{\partial V}{\partial r}$ does not, and it can easily be seen that $\frac{V}{r}$ does not in general; so we have the interesting fact that the solution V_1 is the sum of two functions of V (itself a solution) neither of which is a solution. We can at least give a physical interpretation to the method of finding

a solution of (1) represented by the mathematical operation $\frac{1}{r}\frac{\partial(Vr)}{\partial r}$, where Vr is a solution of (2) and V itself a solution of (1); we have but to add to the doublet of this V as defined above a heating at each point r, which is V divided by the value of r at the point.

The meaning of $\int V\,dr$, where V is a solution of the differential equation, is now plain. It simply means finding a new function of r and t, V^1, whose doublet is the solution V. That is, $\frac{\partial V^1}{\partial r} = V$, and $V^1 = \int V\,dr$. This is subject to the same limitations as before, that the differential equation for V must have its coefficients independent of r, in order that V^1 may be a solution of the equation.

Similarly for equation (2); we have a solution, Vr, to find the meaning of the new solution, V^1r, which we get on performing the integration $\int Vr\,dr$. Since $\frac{\partial(V^1r)}{\partial r} = Vr$, or $\frac{1}{r}\frac{\partial(V^1r)}{\partial r} = V$, we are but finding the distribution of temperature, V^1, whose doublet added to the heating $\frac{V^1}{r}$ gives the distribution of temperature, V, which we started with.

We thus see that (2) has the great advantage over (1) that when we find a solution of the former we can differentiate and integrate it with regard to r for new solutions, but we cannot do so with the latter.

The meaning of $\frac{\partial V}{\partial t}$ and of $\int V dt$ as solutions of (1) are of the same general nature as the similar expressions with r, and are quite evident; we now superpose one heating, $\frac{1}{\Delta t}f(r,t)$ on another, $-\frac{1}{\Delta t}f(r,t)$, after a small interval of time Δt, which we make smaller and smaller indefinitely. We might call this a *time* doublet and the former a *space* doublet. Both $\frac{\partial V}{\partial t}$ and $\int V dt$ are solutions of (1) because the coefficients do not contain t. The same remarks apply to (2) as regards Vr, with the explanations of the former paragraph added. Here equation (2) possesses no advantage over (1).

The meaning of a Fourier's integral may now be given. A solution of (3) for the flow of heat in one dimension is evidently

$V = e^{-ka^2t} \cos \beta(a-x)$, where a and β are arbitrary constants, for it is made up of $V = Ae^{-ka^2t} \cos ax$ and $V = Be^{-ka^2t} \sin ax$, both of which are solutions of (3) as shown above. This equation denotes a distribution of temperatures which has maxima and minima values, the latter being at certain fixed points given by the equation $x = a - (2n + 1)\frac{\pi}{2\beta}$. In general it is very similar to the distribution represented by (4) already studied. $V_1 = V\varphi(a)$ is also a solution, where the temperatures are as before except that they are increased by multiplying every one by $\varphi(a)$, an arbitrary constant function of a. Another solution is got, as described before, by superposing all the heatings formed on reducing the temperatures in V_1 by multiplying each by the very small quantity da, and giving a all values from $-\infty$ to $+\infty$, and then taking the limiting case where da tends to zero. Call this new solution V_2; then $V_2 = \int_{-\infty}^{+\infty} e^{-ka^2t} \cos \beta(a-x)\varphi(a)da$. Repeat this last operation with regard to β; that is, take the distribution of temperatures represented by V_2 and reduce the numerical value of each by multiplying by $d\beta$, then superpose all such heatings formed by giving β every value from 0 to ∞, and finally take the limiting case where $d\beta$ tends to zero. Call this new solution V_3; then $V_3 = \int_0^{\infty} d\beta \int_{-\infty}^{+\infty} e^{-ka^2t} \cos \beta(a-x)\varphi(a)da$. Still another solution is got by reducing every temperature in V_3 in the ratio of π to 1. Call this solution V_4; then $V_4 = \frac{1}{\pi}\int_0^{\infty} d\beta \int_{-\infty}^{+\infty} e^{-ka^2t} \cos \beta(a-x)\varphi(a)da$; it has the special importance and peculiarity, as was first shown by Fourier, that at time zero the distribution of temperature it represents is the same function of x, $\varphi(x)$, that we took originally of a. Similarly every Fourier integral may be interpreted.

Returning now to equation (6) and the curves drawn for it, we can find new solutions by addition; at each point r let us add the temperature for that point and all other points farther from the centre, even to infinity, but first reduced in absolute value by multiplying each by the small quantity dr, which we make ultimately tend to zero. We have but to add on Plate XXIV for any

abscissa (time) the ordinates of all possible curves such as B^1, B^2, etc., below any given one, after reducing them as described. For $t = 0$ and $r = 0$ we would get $(\infty + 0 + 0 + \text{etc.})\, dr$, which as dr diminishes indefinitely gives us some finite value; for other values of r we would get $(0 + 0 + 0 + \text{etc.})\, dr$, which is zero. From the way the curves tend to become parallel it is suggested, and by trial we find, that for $r = 0$ and any finite value of the time not zero the sum of all the ordinates would be constant. We have then the promise of another simple solution, and can foretell its type somewhat, of the form

$$Vr = \int_r^\infty \frac{A}{\sqrt{kt}} e^{-\frac{r^2}{4kt}} dr = \frac{2}{\sqrt{\pi}} B \int_{\frac{r}{2\sqrt{kt}}}^\infty e^{-\beta^2} d\beta, \ldots\ldots (7)$$

where B is an arbitrary constant. On studying this equation we find that Vr at the origin has initially the value B, and maintains that value; at all other points it is initially zero and rises asymptotically with time toward the value B. V itself would be always infinite at the origin and initially zero elsewhere. For the case of linear flow equation (7) represents an infinite plane kept at temperature B in an infinite medium initially zero in temperature.

We can get the solution for an infinitely hot point put into an infinite medium initially zero and left to cool as follows: At time zero apply to the medium the state of temperatures represented by (7) with every temperature increased by multiplying it by the large quantity $\frac{1}{\Delta}$; after time Δt apply also the state of temperatures represented by (7) with sign changed and increased numerically as before; finally make Δt tend to zero. We have seen above that this is equivalent to performing the mathematical operation of differentiation of (7) with regard to t, that is, taking the time doublet of Vr. The reason that this solution is the one required is that the superposition of the two heatings gives Vr a large value at the origin at first and everywhere else a zero value, and then instantaneously makes Vr zero at the origin; that is, at the origin V is initially infinite in temperature and then falls off indefinitely, while all other points begin at zero and rise gradually. These were the conditions we wanted. Hence we have the solution

$$Vr = \frac{\partial}{\partial t}\left[\frac{2}{\sqrt{\pi}} B \int_{\frac{r}{2\sqrt{kt}}}^{\infty} e^{-\beta^2} d\beta\right] = \frac{Er}{(kt)^{\frac{3}{2}}} e^{-\frac{r^2}{4kt}}, \quad \text{..(8)}$$

and

$$V = \frac{E}{(kt)^{\frac{3}{2}}} e^{-\frac{r^2}{4kt}}, \quad \text{.............................. (9)}$$

where E is an arbitrary constant.

Further light can be thrown on this problem by arriving at equation (8) by other methods. Remembering that equation (6) gave Vr initially infinite at the centre and zero elsewhere, and falling in value at the centre and gradually rising to a maximum elsewhere, we see that by taking the space doublet of this Vr we get Vr at the origin first infinite and then zero; that is, V at the origin is at first infinite and then gradually falls off, and is initially zero elsewhere and rises with time. These are the conditions required. Hence the solution is

$$Vr = \frac{\partial}{\partial r}\left[\frac{A}{\sqrt{kt}} e^{-\frac{r^2}{4kt}}\right] = \frac{Er}{(kt)^{\frac{3}{2}}} e^{-\frac{r^2}{4kt}} \quad \text{......(10)}$$

Or we can look at it in this way: We saw that Vr in (6) had exactly the set of values we want V to have in the problem proposed, and the form of the right-hand member of (6), containing as it does r in the factor $e^{-\frac{r^2}{4kt}}$ only, suggests at once that we can get the desired value of V by a simple differentiation with regard to r. This is what we have just done with a good physical reason for the operation.

Or another method. We saw that equation (6) for the case of flow in one direction only was that of an infinitely hot plane cooling in an infinite medium initially zero in temperature, and to get the solution for the similar problem in three dimensions we have but to multiply that solution by two similar ones with y and z substituted for x. This gives

$$V = E \frac{1}{(kt)^{\frac{3}{2}}} e^{-\frac{x^2}{4kt}} e^{-\frac{y^2}{4kt}} e^{-\frac{z^2}{4kt}} = \frac{E}{(kt)^{\frac{3}{2}}} e^{-\frac{r^2}{4kt}} \quad \text{......(11)}$$

The rate of cooling is given by the equation

$$\frac{dV}{dt} = \frac{E}{2k^{\frac{3}{2}}t^{\frac{5}{2}}}\left(\frac{r^2}{2kt} - 3\right)e^{-\frac{r^2}{4kt}}.$$

Each point of the mass not the centre begins by being zero in temperature, then rises to a maximum after a time $t = \frac{r^2}{6k}$, and after this falls off indefinitely toward zero. The forms of the curves given by (9) are exhibited on Plates XXV and XXVI. With values of r as abscissæ curves I' to IV' are for values of the time $\frac{1}{16k}$, $\frac{1}{8k}$, $\frac{1}{4k}$, and $\frac{1}{2k}$ respectively; with values of $4kt$ as abscissæ curves 1' to 5' are for values of the distance 0, $\frac{1}{2}$, 1, $\frac{3}{2}$ and 2 respectively.

The meaning of the constant E is determined by finding the amount of heat supplied initially to the hot point. We have

$$Q = \int\int\int CDV dx\,dy\,dz = \frac{4\pi CDE}{(kt)^{\frac{3}{2}}}\int_0^{\infty} e^{-\frac{r^2}{4kt}} r^2 dr = 8CDE\pi^{\frac{3}{2}} \ldots (12)$$

If we take as our unit of heat that required to raise the mass in a unit of volume of the substance 1°, the total quantity of heat, σ, in these units is

$$\sigma = 8E\pi^{\frac{3}{2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

We could also get the total heat by taking the integral

$\int_0^{\infty} - K\frac{\partial V}{\partial r}4\pi r^2 dt.$ We get from (12) and (13) our equation (11)

in the form

$$V = \frac{Q}{8CD(\pi kt)^{\frac{3}{2}}}e^{-\frac{r^2}{4kt}} = \frac{\sigma}{8(\pi kt)^{\frac{3}{2}}}e^{-\frac{r^2}{4kt}} \ldots\ldots\ldots (14)$$

(See Kelvin's Papers, Vol. II, p. 44.)

We cannot build up by summation the solution for the case of a body of finite dimensions from the above solution for a mathematical point. We wish to pass to a case which has a physical significance, namely, a finitely hot particle left to cool in an infinite

medium of temperature initially zero. We can get a close approximation to this problem by putting the same quantity of heat, σ, into a particle of volume Δv which we put into the mathematical point, and assuming that the state of temperature produced in the surrounding medium is the same as that due to the infinitely hot point and is given accordingly by (14). This equation will represent the real state the better the longer the time which has elapsed, in accordance with the fact emphasized by Fourier that the initial heating is of less and less importance as the time is prolonged. The closeness of the approximation for any given time and distance will be brought out later.

Let the quantity of heat supplied raise the volume Δv to the temperature V_0; then $Q = CDV_0\Delta v$, or $\sigma = V_0\Delta v$; and (14) becomes

$$V = \frac{V_0\Delta v}{8(\pi kt)^{\frac{3}{2}}} e^{-\frac{r^2}{4kt}} \qquad \text{(15)}$$

If the volume Δv is in the form of a sphere of radius R, (15) becomes

$$V = \frac{V_0 R^3}{6\sqrt{\pi}} \frac{1}{(kt)^{\frac{3}{2}}} e^{-\frac{r^2}{4kt}}, \qquad \text{(16)}$$

and it is really for this form of the equation, with R taken as the unit of length, that the curves referred to on Plates XXV and XXVI were drawn. They are, as said, approximations only to the true curves. The latter may be found by the aid of a Fourier's integral. We know that the solution of (2) subject to the condition $V = f(r)$ when $t = 0$ is

$$Vr = \frac{1}{\sqrt{\pi}}\left[\int_{-\frac{r}{2\sqrt{kt}}}^{\infty} (r + 2\sqrt{kt}\,\gamma) f(r + 2\sqrt{kt}\,\gamma) e^{-\gamma^2} d\gamma - \right.$$

$$\left. \int_{\frac{r}{2\sqrt{kt}}}^{\infty} (-r + 2\sqrt{kt}\,\gamma) f(-r + 2\sqrt{kt}\,\gamma) e^{-\gamma^2} d\gamma \right] \quad \text{(17)}$$

Giving $f(r)$ the value V_0 from $r = 0$ to $r = R$, and the value 0 from $r = R$ to $r = \infty$, (17) takes the form

$$V = \frac{V_0}{\sqrt{\pi}} \left[\int_{\frac{r-R}{2\sqrt{kt}}}^{\frac{r+R}{2\sqrt{kt}}} e^{-\gamma^2} d\gamma - \frac{\sqrt{kt}}{r} \left\{ e^{-\frac{(r-R)^2}{4kt}} - e^{-\frac{(r+R)^2}{4kt}} \right\} \right] \quad (18)$$

This then is the exact equation for a sphere of any size of initial temperature V_0 put into an infinite medium of the same material as the sphere of initial temperature zero and left to cool there. The forms of the curves given by this equation are exhibited on Plates XXV and XXVI, along with those of the approximate equation (16). Curves I to IV correspond to I[1] to IV[1], and curves 1 to 5 correspond to 1[1] to 5[1].

We can get an approximate form from equation (18) by expanding it in erms of R; we find

$$V = \frac{V_0 R^3}{6\sqrt{\pi}} \frac{1}{(kt)^{\frac{3}{2}}} e^{-\frac{r^2}{4kt}} \left[1 + \frac{\frac{r^2}{kt} - 6}{40} \frac{R^2}{kt} \right] \ldots\ldots (19)$$

The first term of this is the same as equation (16), found otherwise. Equation (19) gives us a second approximation, and the second term within the bracket will enable us to determine the closeness of (16) as an approximation. In a similar problem, Fourier (Freeman's translation, p. 380) gives a limit to the time when the approximation may be used, but he does not give any means of telling how great the error is in general, and it was for the purpose of bringing this out distinctly that equation (19) and the curves on Plates XXV and XXVI were produced. From Plate XXV we see that the approximate curves are at first steeper and afterward flatter than the exact curves; they make the temperatures too high for points nearer the origin than a certain distance, and too low for points farther away. Indeed curves I and I[1] are very little alike for any value of r. As the value of the time for which the curve is drawn is taken greater and greater the curves approach each other more and more nearly, even for points less distant than unity (which are inside the little sphere), for which we might have expected little agreement. This makes evident the fact to which Fourier calls attention at the place just cited; one is very apt to assume that the curves would approach each other more and more as r is taken greater and greater, no matter what the value of t; but just the reverse is true,

the curves approach each other more and more for greater and greater values of the time, no matter what the distance. This is seen more distinctly from an examination of Plate XXVI. There it will be seen also that the approximate curves are slower in reaching their maximum values, as well as that they have different maxima. For distances less than unity the approximate curves start at ∞, while the exact curves start at $V = V_0$; for the distance unity the exact curve starts abruptly at $\frac{V_0}{2}$, while the approximate curve starts at 0 then gradually rises and has a maximum value less than $\frac{V_0}{3}$. For distances greater than unity both curves start at the origin.

From an inspection of the second term of (19) we can foretell the approximate accuracy of (16). Taking R as the unit of length, if $kt < 15$ the error in the value of $\frac{V}{V_0}$ will be everywhere greater than 1% except in the immediate neighborhood of $r = \sqrt{6kt}$, at which point the error is practically zero. For instance, for $kt = \frac{1}{2}$ (curves IV and IV') the approximate curve is 33% too high at $r = 0$, 22% at $r = 1$, correct at about 1.8, and 38% too low at 3. If $kt = 15$, the error is not more than 1% from $r = 0$ to $r = 13.4$. If $kt = 25$ the error is not more than 1% from $r = 0$ to $r = 20$. In general, for any value of kt the error is not more than 1% from $r = 0$ to $r = \sqrt{6kt + \frac{2}{5}(kt)^2}$, and from $r = 0$ to $r = \sqrt{6kt}$ the error decreases gradually from $\frac{15}{kt}$% to zero, and after that increases again. If we want results accurate to .01%, kt must be at least 1500, and in general for any value of kt greater than this the error is not more than .01% from $r = 0$ to $r = \sqrt{6kt + \frac{1}{250}(kt)^2}$, and from $r = 0$ to $r = \sqrt{6kt}$ the error decreases gradually from $\frac{15}{kt}$% to zero, and after that increases again.

From equation (15) we can build up by summation the equation for the case of a body of any shape or size initially at V_0 cooling in an infinite medium initially zero. In order to bring out a very interesting difference between summation and integration we shall apply equation (15) to the case of an infinite space, one-half of which is initially at V_0 and the other half at zero, the two parts being separated by an infinite plane surface. We shall first have to find the solution for a plane lamina. Take the central plane of the lamina as the plane of yz, and the origin where a perpendicular

from the point P, at which we want to know the temperature, meets this plane. Call the length of this perpendicular x. Break up the lamina into concentric rings of radius ρ about this origin, and let the distance of every point in one of such rings from the point P be r and the thickness of the lamina Δx; then we have

$$V = \frac{V_0}{8(\pi kt)^{\frac{3}{2}}}\int_0^{\infty} e^{-\frac{x^2+\rho^2}{4kt}} 2\pi\rho.\Delta x.d\rho = \frac{V_0\Delta x}{2(\pi kt)^{\frac{1}{2}}} e^{-\frac{x^2}{4kt}} \quad \ldots\ldots\ldots (20)$$

From the symmetry of the problem this is evidently a case of linear flow, and the solution must satisfy equation (3). Knowing this solution (we can get it otherwise), the solution for three dimensions given in (15) can be deduced; we have but to multiply the value of $\frac{V}{V_0}$ for the case of one dimension by two similar expressions with y and z respectively substituted for x.

The corresponding electrical problem is that of an infinite cable with no lateral loss by leakage touched for an instant to a condenser of potential V_0. If there is lateral leakage equation (20) is still the solution of the electrical problem; V is then not the potential, but the potential can be derived easily from it, as is well known.

If Q or σ, according to the unit of heat used, is the amount of heat required to raise the mass of a section of the plate of unit area by V_0 degrees, then $Q = CDV_0\Delta x$, or $\sigma = V_0\Delta x$, and equation (20) becomes

$$V = \frac{Q}{2CD(\pi kt)^{\frac{1}{2}}} e^{-\frac{x^2}{4kt}} = \frac{\sigma}{2(\pi kt)^{\frac{1}{2}}} e^{-\frac{x^2}{4kt}} \quad \ldots\ldots\ldots\ldots (21)$$

Of course this equation is of only the same grade of approximation as (15). It will be the more nearly exact the smaller Δx and, since the product of V_0 and Δx measures the heat in a section of unit area and is to remain constant, the greater V_0. In the limit we should have the solution for an infinitely hot plane. The form of this solution we have already found; it is from (6) and the remarks following it

$$V = \frac{A}{\sqrt{kt}} e^{-\frac{x^2}{4kt}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

Calling Q the total heat associated initially with a unit of area of

the plate, we find $Q = 2\int_0^\infty CDV\,dx = 2ACD\sqrt{\pi}$; and this value of A reduces (22) to the form (21). Hence the general form of equation (21), which is approximate for a plate of actual thickness Δx, is exact for the infinitely hot plane. We shall revert to this important fact later.

If we want the exact equation for the plate of thickness Δx we can get it by the use of a Fourier integral. Making the obvious changes in (17) to suit it to the case of linear flow, and giving $f(x)$ the value V_0 from $x = -\frac{\Delta x}{2}$ to $x = \frac{\Delta x}{2}$ and the value 0 for all other values of x, we find

$$V = \frac{V_0}{\sqrt{\pi}} \int_{-\frac{x - \frac{\Delta x}{2}}{2\sqrt{kt}}}^{-\frac{x - \frac{\Delta x}{2}}{2\sqrt{kt}}} e^{-\gamma^2}\, d\gamma \ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

Putting this in an approximate form, we have

$$V = \frac{V_0 \Delta x}{2(\pi kt)^{\frac{1}{2}}} e^{-\frac{x^2}{4kt}} \left[1 + \frac{\frac{x^2}{kt} - 2}{96} \frac{(\Delta x)^2}{kt}\right], \ldots\ldots (24)$$

the first term of which is equation (20). The forms of the curves for (20) are exhibited on Plates XXIII and XXIV. With values of x as abscissæ curves A^1 to A^4 are for values of the time $\frac{1}{16k}$, $\frac{1}{8k}$, $\frac{1}{4k}$ and $\frac{1}{2k}$ respectively; with values of $4kt$ as abscissæ curves B^1 to B^5 are for values of the distance 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$ and 1 respectively. The second term of (24) enables us to tell approximately the degree of closeness of (20) to the exact equation (23). Taking Δx as the unit of length, if $kt < \frac{25}{12}$ the error will be everywhere greater than 1% except in the neighborhood of $x = \sqrt{2kt}$ where it is practically zero. If $kt = \frac{25}{12}$ the error is not more than 1% from $x = 0$ to $x = 2.9$, being 1% too high at $x = 0$, zero at $x = 2$, and 1% too low at $x = 2.9$. If $kt = 25$ the error is $\frac{1}{12}$% too high at $x = 0$, zero at 7, and 1% too low at 26. This is then a nearer approximation than the one discussed for the case of a hot particle, as was to

be expected. In general, for any value of kt the error is not more than 1% from $x = 0$ to $x = \sqrt{2kt + \frac{24}{25}(kt)^2}$, and for any value of kt greater than $\frac{2500}{12}$ the error is not more than .01% from $x = 0$ to $x = \sqrt{2kt + \frac{24}{2500}(kt)^2}$; from $x = 0$ to $x = \sqrt{2kt}$ the error decreases gradually from $\frac{25}{12kt}$% to zero, and after that increases again.

The correspondingly approximate equation for the current or flow of heat in this case is

$$I = -K\frac{\partial V}{\partial x} = \frac{KV_0\Delta x}{4\sqrt{\pi}}\frac{x}{(kt)^{\frac{3}{2}}}e^{-\frac{x^2}{4kt}} = \frac{K\sigma}{4\sqrt{\pi}}\frac{x}{(kt)^{\frac{3}{2}}}e^{-\frac{x^2}{4kt}} \quad \ldots.(25)$$

The forms of these curves are given on Plates XXVII and XXVIII. With values of x as abscissæ curves C^1 and C_1^1, C^2 and C_1^2, and C_1^3 are for values of the time $\frac{1}{16k}$, $\frac{1}{8k}$ and $\frac{1}{4k}$ respectively; with values of $4kt$ as abscissæ curves D^1 and D_1^1, D^2 and D_1^2, and D_1^3 are for value of the distance $\frac{1}{4}$, $\frac{1}{2}$ and 1 respectively.

The exact equation for the flow, found from (23), is

$$I = \frac{KV_0}{2(\pi kt)^{\frac{1}{2}}}\left[e^{-\frac{(x-\frac{1}{2}\Delta x)^2}{4kt}} - e^{-\frac{(x+\frac{1}{2}\Delta x)^2}{4kt}}\right], \quad \ldots\ldots(26)$$

the curves for which have not been drawn.

By adding up the effects of an infinite number of such plates we can get the temperature due to one-half of space initially at a uniform temperature V_0 and the other half at zero temperature. Take the point P, at which the temperature is desired, in the cold half and at a distance x from the surface of separation, and take the origin in that surface at the foot of the perpendicular from P. Let one of the plates making up the other half of the medium be distant ξ from the origin. Then the x of equation (20) becomes $x + \xi$, and Δx becomes $\Delta\xi$; hence the temperature at P due to a series of such plates extending from $\xi = 0$ to $\xi = \infty$, as found by integration, is

$$V = \frac{V_0}{2(\pi kt)^{\frac{1}{2}}}\int_0^\infty e^{-\frac{(x+\xi)^2}{4kt}}d\xi = \frac{V_0}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{kt}}}^{\infty} e^{-\beta^2}d\beta$$

$$= \frac{V_0}{2}\left[1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{kt}}} e^{-\beta^2}\,d\beta\right] \ldots\ldots\ldots\ldots (27)$$

We could arrive at the solution for this case by using Fourier's integrals, as we did for equation (23), giving $f(x)$ the value V_0 from $x = -\infty$ to $x = 0$ and the value zero from $x = 0$ to $x = \infty$. We get at once equation (27) again.

This latter method gives the exact solution for the problem and yet it gives the same result as the former method, from which one might expect naturally enough an approximate solution, since we get it by integrating solutions that were approximate. This is the point to which attention was called in applying our results to this case; we have the integration of approximate solutions an exact solution. The first explanation offered of this unexpected result is apt to be that the approximation used is the more exact as the distance $x + \xi$ is the greater; but we have seen earlier that just the contrary is true and that at great distances (20) ceases to be properly called a solution unless the time is taken very great. The real explanation is simply that the operations of summation and integration are not always the same, and this is a case in point. Nothing is commoner in applying mathematics to physics than to use mathematical processes with laxity and to test the legitimacy of the application by the results. It is so uncommon to have a summation made improperly by integration that we lose sight of the mathematical fact that the operations are not equivalent. We take similarly the first two terms of a Taylor's series expansion as a sufficiently close approximation in almost any piece of analysis, without questioning whether the function under consideration can be so expanded and without reference to the value of the terms disregarded; we take differential coefficients without asking whether they can have a meaning, etc. The good excuse offered is that the chances are overwhelmingly in our favor, and that if we have made a mistake we shall quickly find it out from the results. Had we actually made a summation in the above problem we should have got an approximate result, but by integrating we get the limit toward which the summation tends as $d\xi$ tends towards zero, and it happens in this case that this is the exact solution. In finding an area we take a series of strips of area of ydx and however infinitesimally small dx is, so long as it is something and not zero, the sum

of such strips is not the exact area required; $\int ydx$ is the limit toward which the sum tends as dx tends to zero, and we know from the familiar example of Fourier's series how the value can change actually in the limit. It happens in the present case that as $d\xi$ is made smaller and smaller, and V_0 correspondingly greater and greater in order to keep σ constant, in the limit $\frac{\sigma}{2(\pi kt)^{\frac{1}{2}}} e^{-\frac{x^2}{4kt}}$ is the exact solution for an infinite plane (see under (21) and (22)). So in making the integration above, that is, in finding the limit of the summation, we get necessarily an exact solution because in the limit each term of the solution is exact. Had we approached the limit in some other way than in keeping σ constant we might have got quite a different result.

The forms of the curves for (27) are shown on Plates XXVII and XXVIII. Curves E^1, E^2 and E^3 are drawn with values of x as abscissæ for values of the time $\frac{1}{16k}, \frac{1}{4k}$ and $\frac{1}{k}$ respectively; curves F^1, F^2 and F^3 are drawn with values of $4kt$ as abscissæ for values of the distance $\frac{1}{4}$, $\frac{1}{2}$, and 1 respectively.

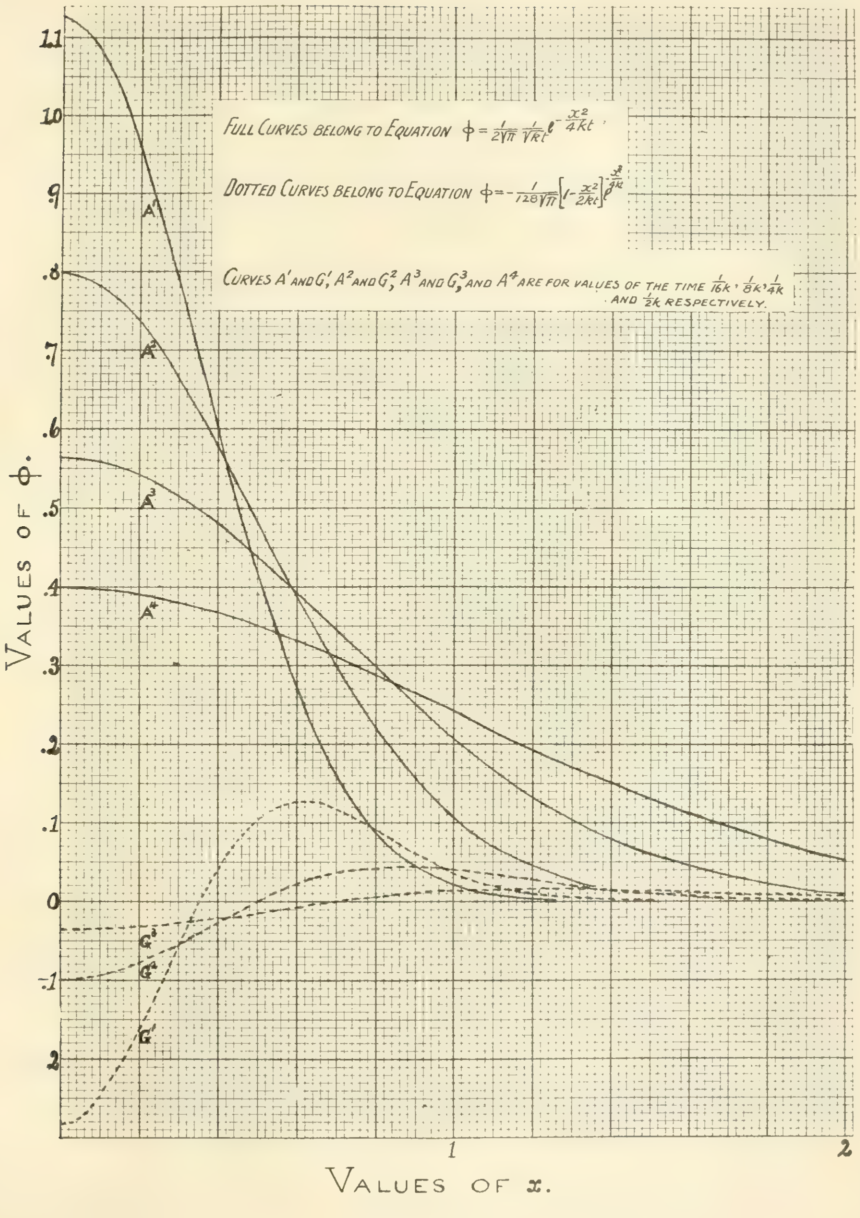
Since the current or flow is got from the temperature by a differentiation with regard to x, and since equation (27) was got from (20) by an integration with regard to x, it is evident that the curves for the potential or temperature in (20) are the curves for current in the present problem.

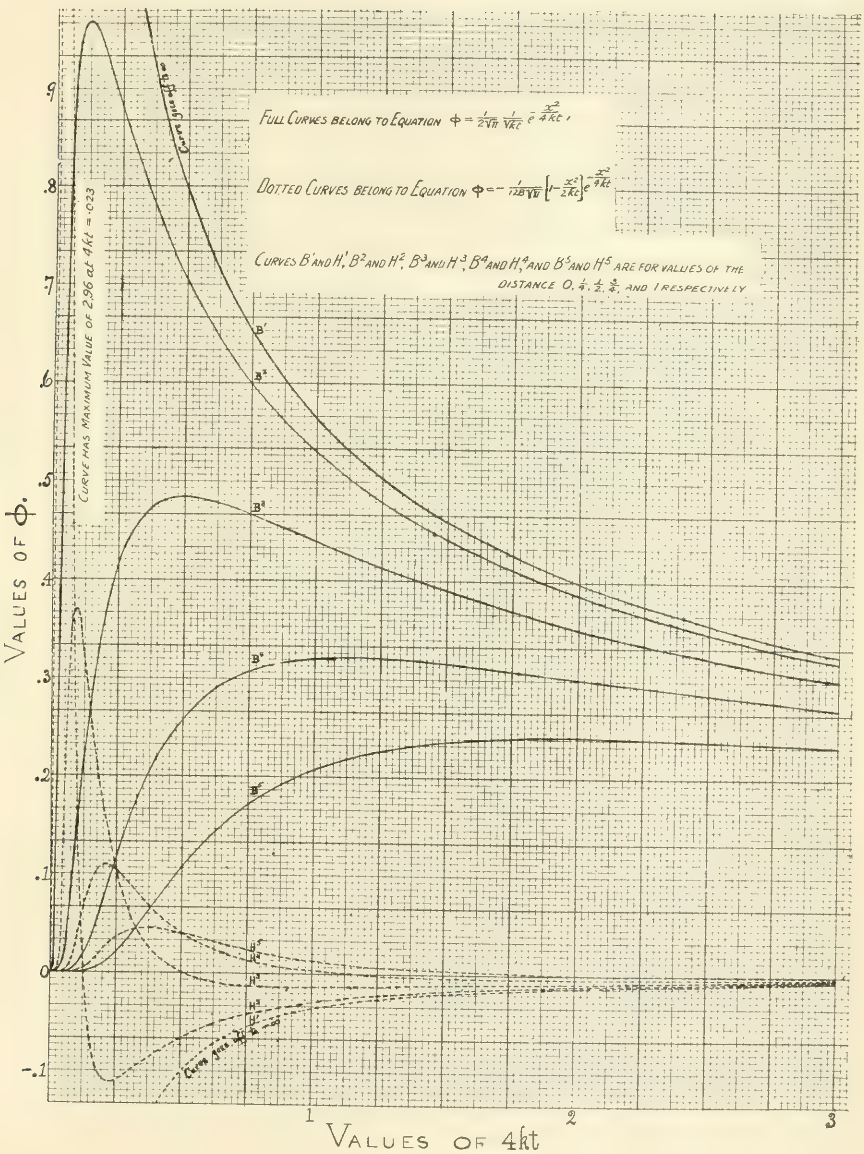
$$I = -K\frac{\partial V}{\partial x} = \frac{KV_0}{2(\pi kt)^{\frac{1}{2}}} e^{-\frac{x^2}{4kt}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (28)$$

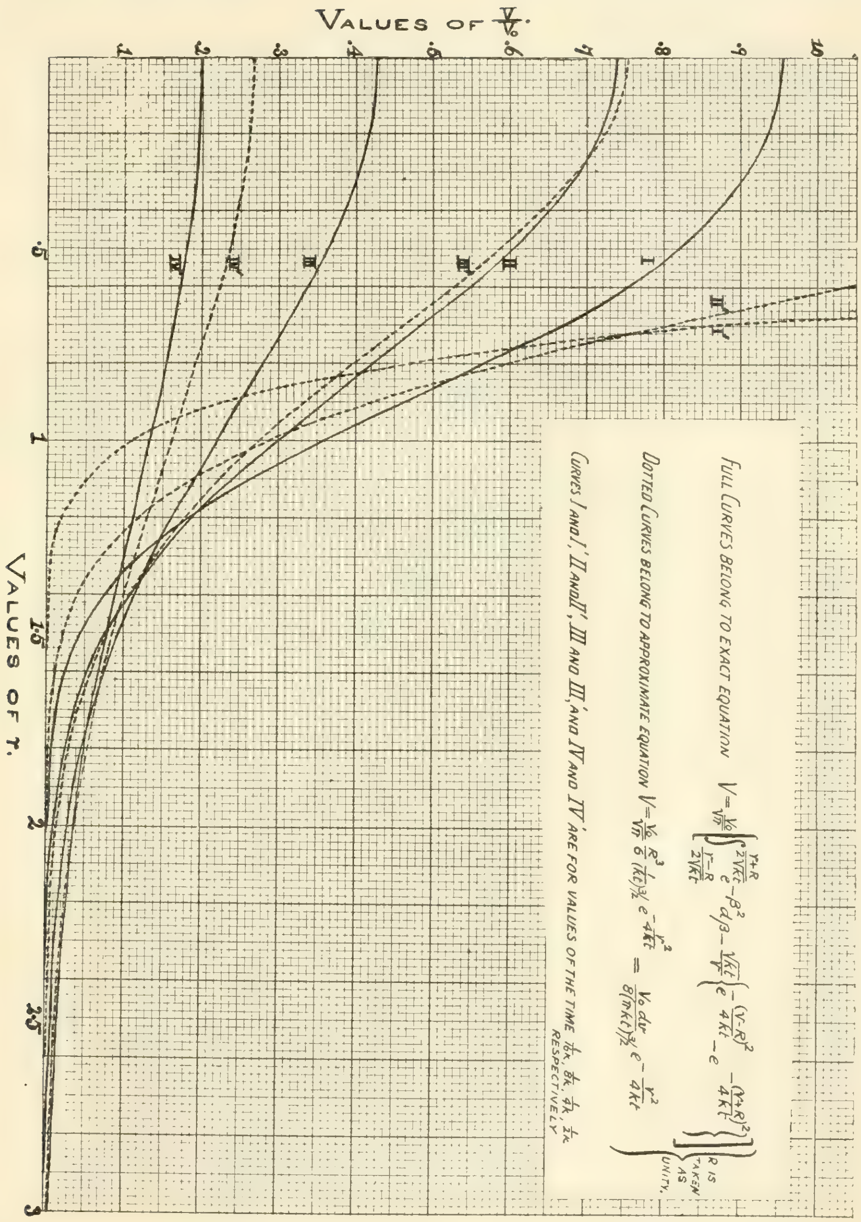
These curves are given on Plates XXIII and XXIV for points to the right of the origin; the form for points to the left is obvious, since the curves are symmetrical about the yz plane.

PHYSICAL LABORATORY, BRYN MAWR COLLEGE.

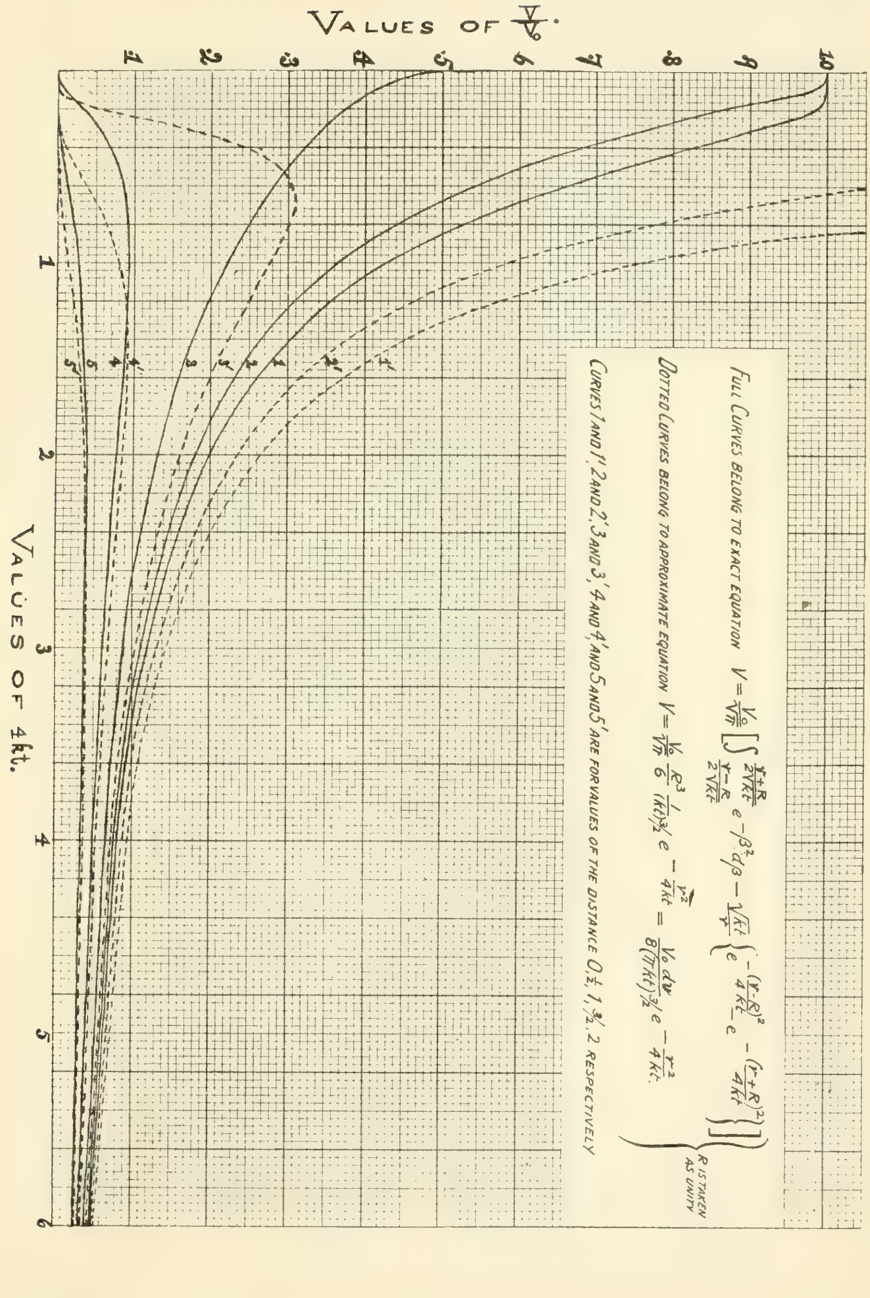
April 3, 1902.


1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
-.1
.2
1
2
VALUES OF ϕ.
VALUES OF x.
A^1
A^2
A^3
A^4
G^1
G^2
G^3
FULL CURVES BELONG TO EQUATION $\phi = \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{kt}} e^{-\frac{x^2}{4kt}}$,
DOTTED CURVES BELONG TO EQUATION $\phi = -\frac{1}{128\sqrt{\pi}} \left[1 - \frac{x^2}{2kt}\right] e^{-\frac{x^2}{4kt}}$
CURVES A^1 AND G^1, A^2 AND G^2, A^3 AND G^3, AND A^4 ARE FOR VALUES OF THE TIME $\frac{1}{16k}$, $\frac{1}{8k}$, $\frac{1}{4k}$ AND $\frac{1}{2k}$ RESPECTIVELY.

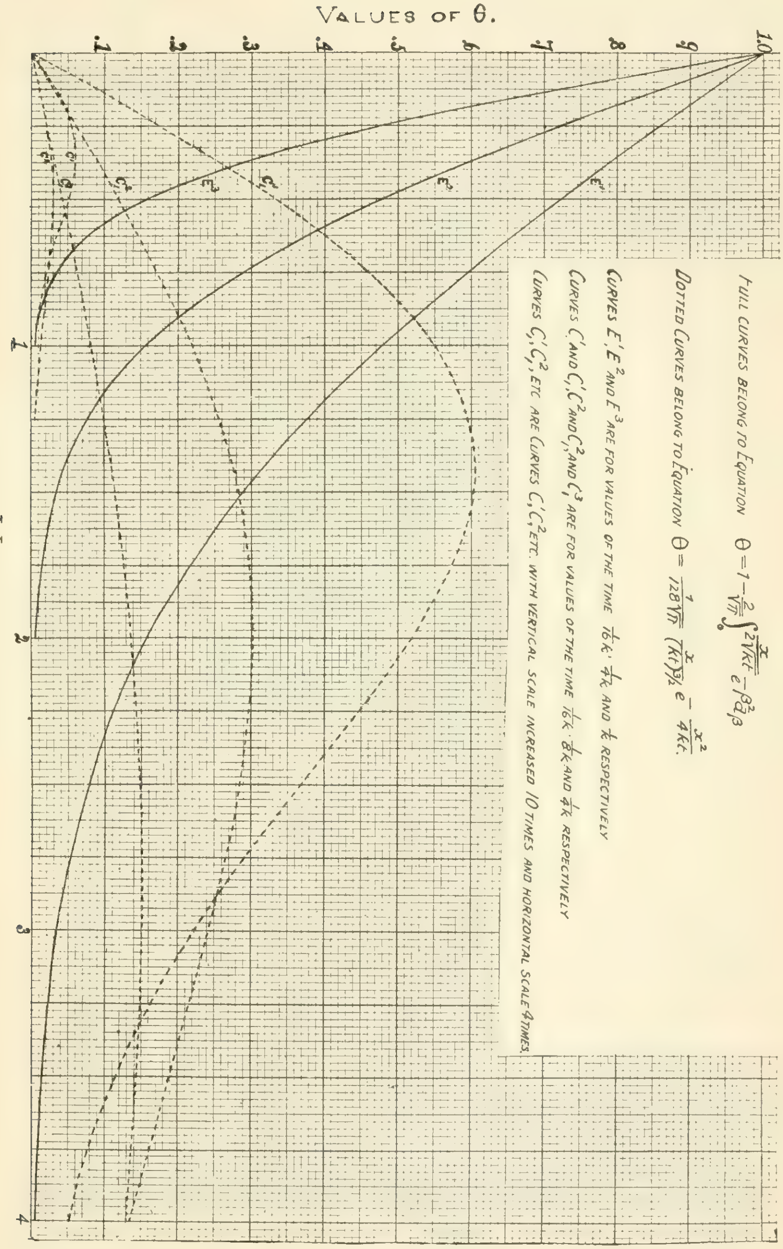

Full Curves belong to Equation $\phi = \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{kt}} e^{-\frac{x^2}{4kt}}$,
Dotted Curves belong to Equation $\phi = -\frac{1}{128\sqrt{\pi}} \left[1 - \frac{x^2}{2kt}\right] e^{-\frac{x^2}{4kt}}$
Curves B^1 and H^1, B^2 and H^2, B^3 and H^3, B^4 and H^4, and B^5 and H^5 are for values of the distance 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1 respectively
Curve has maximum value of 2.96 at 4kt = .023
Values of ϕ.
Values of 4kt

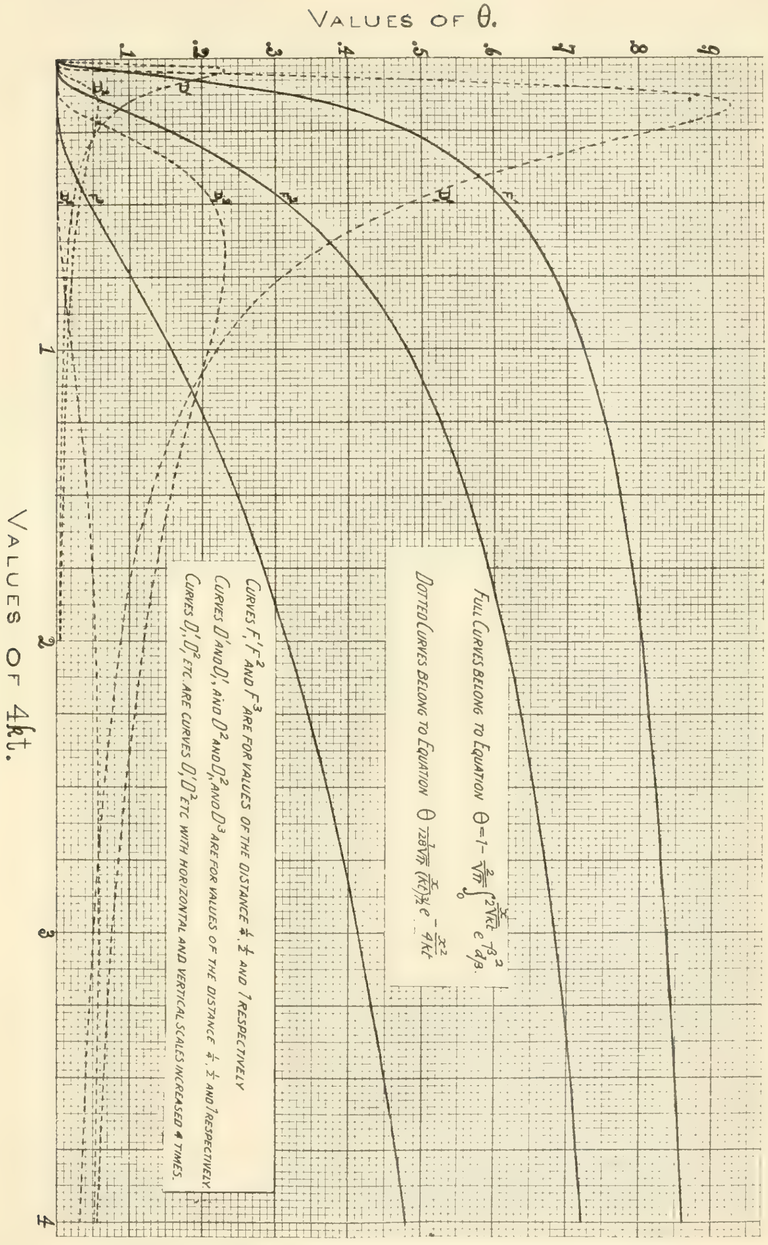


VALUES OF $\frac{V}{V_0}$.
.1 .2 .3 .4 .5 .6 .7 .8 .9 1.0
VALUES OF r.
.5 1 1.5 2 2.5 3
I I′ II II′ III III′ IV IV′
FULL CURVES BELONG TO EXACT EQUATION $V=\frac{V_0}{\sqrt{\pi}}\left[\int_{\frac{r-R}{2\sqrt{kt}}}^{\frac{r+R}{2\sqrt{kt}}} e^{-\beta^2}d\beta-\frac{\sqrt{kt}}{r}\left\{e^{-\frac{(r-R)^2}{4kt}}-e^{-\frac{(r+R)^2}{4kt}}\right\}\right]$
DOTTED CURVES BELONG TO APPROXIMATE EQUATION $V=\frac{V_0}{\sqrt{\pi}}\frac{R^3}{6}\frac{1}{(kt)^{3/2}}e^{-\frac{r^2}{4kt}} = \frac{V_0\,dv}{8(\pi kt)^{3/2}}e^{-\frac{r^2}{4kt}}$
R IS TAKEN AS UNITY.
CURVES I AND I′, II AND II′, III AND III′, AND IV AND IV′ ARE FOR VALUES OF THE TIME $\frac{1}{16k}$, $\frac{1}{8k}$, $\frac{1}{4k}$, $\frac{1}{2k}$ RESPECTIVELY



Values of $\frac{V}{V_0}$.
.1 .2 .3 .4 .5 .6 .7 .8 .9 1.0
Values of 4kt.
1 2 3 4 5 6
Full Curves belong to exact equation $V = \frac{V_0}{\sqrt{\pi}}\left[\int_{\frac{r-R}{2\sqrt{kt}}}^{\frac{r+R}{2\sqrt{kt}}} e^{-\beta^2} d\beta - \frac{\sqrt{kt}}{r}\left\{e^{-\frac{(r-R)^2}{4kt}} - e^{-\frac{(r+R)^2}{4kt}}\right\}\right]$
Dotted Curves belong to approximate equation $V = \frac{V_0}{\sqrt{\pi}} \frac{R^3}{6} \frac{1}{(kt)^{3/2}} e^{-\frac{r^2}{4kt}} = \frac{V_0\, dv}{8(\pi kt)^{3/2}} e^{-\frac{r^2}{4kt}}$
R is taken as unity
Curves 1 and 1', 2 and 2', 3 and 3', 4 and 4', and 5 and 5' are for values of the distance 0, ½, 1, 3/2, 2 respectively


VALUES OF θ.
.1 .2 .3 .4 .5 .6 .7 .8 .9 1.0
VALUES OF x.
1 2 3 4
E¹ E² E³ C¹ C² C³ C₁¹ C₁² C₁³
FULL CURVES BELONG TO EQUATION $\theta = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{kt}}} e^{-\beta^2} d\beta$
DOTTED CURVES BELONG TO EQUATION $\theta = \frac{1}{128\sqrt{\pi}} \frac{x}{(kt)^{3/2}} e^{-\frac{x^2}{4kt}}$
CURVES E¹, E² AND E³ ARE FOR VALUES OF THE TIME $\frac{1}{16k}$ · $\frac{1}{4k}$ AND $\frac{1}{k}$ RESPECTIVELY
CURVES C¹ AND C₁¹, C² AND C₁², AND C₁³ ARE FOR VALUES OF THE TIME $\frac{1}{16k}$ · $\frac{1}{8k}$ AND $\frac{1}{4k}$ RESPECTIVELY
CURVES C₁¹, C₁², ETC. ARE CURVES C¹, C², ETC. WITH VERTICAL SCALE INCREASED 10 TIMES AND HORIZONTAL SCALE 4 TIMES.


VALUES OF θ.
VALUES OF $4kt$.
FULL CURVES BELONG TO EQUATION $\theta = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{kt}}} e^{-\beta^2} d\beta$.
DOTTED CURVES BELONG TO EQUATION $\theta = \frac{1}{128\sqrt{\pi}} \frac{x}{(kt)^{3/2}} e^{-\frac{x^2}{4kt}}$
CURVES $F^1 F^2$ AND F^3 ARE FOR VALUES OF THE DISTANCE $\frac{1}{4}$, $\frac{1}{2}$ AND 1 RESPECTIVELY
CURVES D^1 AND D_1^1, AND D^2 AND D_1^2, AND D^3 ARE FOR VALUES OF THE DISTANCE $\frac{1}{4}$, $\frac{1}{2}$ AND 1 RESPECTIVELY.
CURVES D_1^1, D_1^2, ETC. ARE CURVES D^1, D^2 ETC WITH HORIZONTAL AND VERTICAL SCALES INCREASED 4 TIMES.

[Reprinted from the PHYSICAL REVIEW, Vol. XV., No. 6, Dec., 1902.]

AN INSTRUMENT FOR DRAWING A SINE CURVE.

BY A. STANLEY MACKENZIE.

THE instrument here described has been found useful in lecture-room demonstration, and in drawing sine curves of various wave-lengths and amplitudes for any practical purpose, such as addition of these curves to show the effect of phase or the approximate construction of the curve representing a Fourier's series, when one is not fortunate enough to possess an elaborate instrument of the Michelson[1] pattern. Mach[2] devised an apparatus intermediate between the two.

The idea underlying the present instrument may be stated very briefly in the following way. Suppose one of a pair of parallel rules to be fixed. Let one of the arms of the rulers be made of a disc with a groove on its circumference to which one end of a string is fastened and upon which the string winds (or unwinds) as the disc turns. When the disc rotates uniformly the free end of the string moves uniformly and its direction of motion can easily be made parallel to the edges of the rulers; at the same time the movable ruler has the up and down component of its motion simple harmonic. On transferring these two motions to a common point, the latter will move in a sine curve.

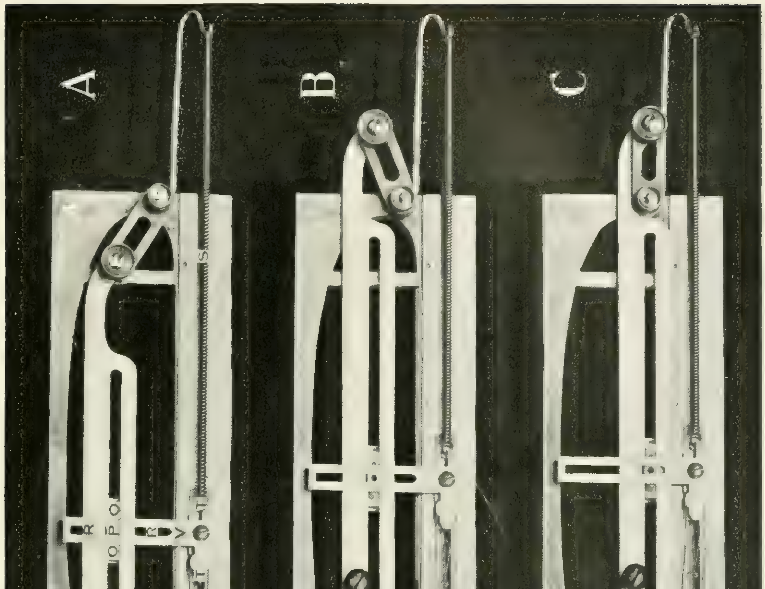
How this is carried out will be evident from an inspection of the accompanying illustrations, A, B, C, which show the instrument in three successive stages of its motion. *AA* is a skeleton base-plate of metal. *C* is the movable ruler, cut away where necessary in order that it may pass the pins at *E*, *E*, *E*. *G*, attached to the base-plate, is the fixed ruler. *EF*, *EF*, *EF* are the arms of the rulers and work on the three fixed pins *E*, *E*, *E*, being secured by binding nuts bearing against shoulders. The length of the arms can be regulated by means of the pins and binding nuts *F*, *F*, *F*, moving

[1] Phil. Mag. [5], 45, 85, 1898.
[2] Pogg. Ann., 129, 464, 1866.

in slots. This determines the amplitude of the simple harmonic motion. The middle arm is a disc; one end of the string passes from the groove on the disc to its face and is there secured under a binding nut; the string after leaving the groove passes round a small pulley M to "the slider" V. The movable ruler has in it a slot, N, whose central line coincides with the line EEE when the rulers touch; in this slot moves the guide OPO whose square central portion, P, projects upwards into the slot, RR, in the slider, and carries underneath it the tracing pencil. In the base-plate is a T-slot, SS, in which moves the guide, TT, to which the slider is rigidly connected. At T is attached one end of a spiral spring (the other end of which is fixed to the base-plate) to keep the string taut. The other end of the guide carries a small pulley and a hook at T. The string may be connected directly to this hook, thus giving the longest wave-length, λ, equal to the circumference of the groove on the disc; or it may pass around the pulley T and be attached to a pin at Y, giving a wave-length $\lambda/2$; or it may pass around the pulleys at T and Y and have its end attached to the hook at T (as in the illustrations), giving a wave-length $\lambda/3$. The whole movement is actuated by a handle at C. The base-plate stands on pointed feet, so that when placed on the sheet of paper or cardboard it may not slip. The size of the base-plate used is 25⅝ by 12⅜ inches. The string is a bass string of a banjo. The curves which the instrument draws after a little practice are very smooth, and the wave-lengths are remarkably exact submultiples of the circumference of the disc. For smaller wave-lengths the arrangement of pulleys could be carried further, or new discs used.

PHYSICAL LABORATORY,
BRYN MAWR COLLEGE, April, 1902.

(To face page 367.)



MACKENZIE.